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ABSTRACT

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The Landau-Ginzburg (LG) B-Model is a significant feature of singularity theory and mirror symmetry. Krawitz in [Kra10], guided by work of Kaufmann, see [Kau06], provided an explicit construction for the LG B-model when using diagonal symmetries of a quasihomogeneous, nondegenerate polynomial. In this thesis we discuss aspects of how to generalize the LG B-model construction to allow for nondiagonal symmetries of a polynomial, and hence nonabelian symmetry groups. The construction is generalized to the level of graded vector space and the multiplication developed up to an unknown factor. We present complete examples of nonabelian LG B-models for the polynomials $x^2y + y^3$, $x^3y + y^4$, and $x^3 + y^3 + z^3 + w^2$.

Keywords: Mirror Symmetry, Singularity Theory, Landau-Ginzburg B-model, Frobenius Algebra

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Contents

List of Tables

List of Figures

Chapter 1. Introduction

In this thesis we present some aspects of an algebraic construction that is significant to mirror symmetry and singularity theory. It is a generalization of an idea originating in singularity theory but found to be significant in mirror symmetry.

Singularity theory is the study of things going bad. For us, that means points of a manifold where the tangent space is not of the expected dimension or points of an algebraic hypersurface where the gradient goes to 0. An algebraic hypersurface is defined as the zero-locus of a polynomial. A *singularity* is a point where the polynomial vanishes and all of its partials vanish. Much of the theory of singularities centers on isolated singularities. Our constructions depend on several theorems which require isolated singularities, so this is a requirement for all of the polynomials we use.

Definition 1.1 (nondegenerate)**.** We call a polynomial W having an isolated singularity at the origin *nondegenerate*.

V.I. Arnold gave an exhaustive classification of singularities which is well-known in the literature. We provide here a sample of some simpler singularity classes.

	Simple Singularities	
A_k	$x^{k+1},\ k \geq 1$	
D_k	$x^2y + y^{k-1},\ k \geq 4$	
E_6	$x^3 + y^4$	
E_7	$x^3 + xy^3$	
E_8	$x^3 + y^5$	
	Parabolical singularities	
P_8	$x^3 + y^3 + z^3 + axyz$	$a^3 + 27 \neq 0$
X_9	$x^4 + y^4 + ax^2y^2$	$a^2 \neq 4$
J_{10}	$x^3 + y^6 + ax^2y^2$	$4a^3 + 27 \neq 0$

Table 1.1: Some of V.I. Arnold's classes of singularities

Since these singularities are of broad interest, we draw our examples from these lists. In particular, we will later investigate D_4 and P_8. In Arnold's classification, he found surprising correspondences between certain pairs of singularities. These singularities and the surprising correspondences are of interest outside of singularity theory; they arise in mirror symmetry as well.

Mirror symmetry is a phenomenon discovered by string theorists in the high-energy physics community. In the 1980s and 1990s, these scientists developed certain field theories in solving the equations of 'string' propagation, or in the study of dynamics modeling particles as strings. The field theories [IV90] depend ultimately on quasihomogeneous, nondegenerate polynomials.

Definition 1.2 (quasihomogeneous)**.** A polynomial $W \in \mathbb{C}[x_1, \ldots, x_n]$ is said to be *quasihomogeneous* if there are rational numbers $q_1, \ldots, q_n$ such that $W(\lambda_1^q x_1, \ldots, \lambda_n^q x_n) = \lambda W$. The q_i are called the *weights* of W.

For example, $P_8 = x^3+y^3+z^3$ (with $a = 0$) has as weights $(1/3, 1/3, 1/3)$, since $(\lambda^{1/3}x)^3 + (\lambda^{1/3}y)^3 + (\lambda^{1/3}z)^3 = \lambda(x^3 + y^3 + z^3)$.

For a given nondegenerate, quasihomogeneous polynomial and choice of symmetry group of the polynomial, there is both a geometric and an algebraic construction, and each of these comes in an $\mathscr{A}$- and a $\mathscr{B}$- model. The geometric constructions involve Calabi-Yau manifolds, while the algebraic constructions are known as Landau-Ginzburg (LG) theories. As Arnold did, the physicists found surprising correspondences; certain pairs of $\mathscr{A}$- and $\mathscr{B}$-models were found to be isomorphic. Additionally, a correspondence was observed between the algebraic and geometric constructions that matched the isomorphisms between the $\mathscr{A}$- and $\mathscr{B}$-models. Mirror symmetry in this context refers to these isomorphisms and correspondences.

Our research group works mostly in the algebraic picture, studying the mirror symmetry between Landau-Ginzburg (LG) $\mathscr{A}$- and $\mathscr{B}$-models. String theorists used singularity theoretic objects such as the Milnor ring to develop the LG $\mathscr{B}$-models in the 1980s and 1990s [BH92, IV90]. While these string theorists had the physical motivation for the constructions, mathematicians like Kaufmann explored them as mathematical objects of interest (see [Kau02, Kau03, Kau06]). While these objects have several layers of structure, the orbifolded Frobenius algebra, or Frobenius algebra with a group action, of the $\mathscr{A}$- and $\mathscr{B}$-models encodes most of the information of the construction. In this thesis, we don't go beyond the level of Frobenius algebra.

Guided by Kaufmann's ideas, Krawitz in [Kra10] developed a LG $\mathscr{B}$-model construction that yielded a Frobenius algebra in the case of diagonal symmetry groups. For certain nice polynomials (see the following remark on invertible polynomials), he demonstrated that his construction was the proper choice for mirror symmetry; i.e. the $\mathscr{A}$- and $\mathscr{B}$-models are isomorphic as vector spaces for a well-understood choice of polynomials and groups.

Remark (Invertible Polynomials). Past results in Landau-Ginzburg mirror symmetry were particularly simple in the case of certain nice polynomials called *invertible polynomials*, which are sums of the following types of polynomials. Many of these appear on Arnold's list and we often use them as a starting point in our investigations. They consist of

- *Fermat*-type polynomials, of the form x^a. These are the A series polynomials from Arnold's list.
- *Loop*-type polynomials, of the form $x_1^{a_1}x_2 + \ldots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_1$
- *Chain*-type polynomials, of the form $x_1^{a_1}x_2 + \ldots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}$. Each polynomial of the D_k series from Arnold's list is a chain.

End of remark

Krawitz' proof extended to the level of vector space for diagonal groups and to the level of Frobenius algebra for trivial $\mathscr{B}$-model groups. Francis, Johnson, Jarvis, and Webb (Suggs), in [FJJS12], demonstrated that Krawitz' construction was the right choice for most invertible polynomials and most groups of diagonal symmetries.

On the $\mathscr{A}$-side, Fan, Jarvis, and Ruan provided a general construction for the LG $\mathscr{A}$-model [FJR13, FJR07, FJR08] in the case of diagonal groups. More recently, Fan, Jarvis and Ruan proved the existence of $\mathscr{A}$-models for arbitrary symmetry groups [FJR15]. A $\mathscr{B}$-model with nonabelian groups is necessary to continue to develop mirror symmetry.

In this thesis we discuss the extension of the Landau-Ginzburg constructions to choices of nonabelian symmetry groups. Kaufmann's thoughts on graded Frobenius algebras with group action are not specific to diagonal symmetry groups; only the construction developed by Krawitz requires this condition on the symmetries. Guided in part by Kaufmann's and Krawitz's ideas, we provide some specific examples of an orbifolded Frobenius algebra (recall this is a Frobenius algebra with group action) where the symmetry group is not abelian. Along the way, we discuss the factor required to make the prospective multiplication associative, for which we do not yet have a general expression. While LG models with diagonal groups do have levels of structure beyond that of Frobenius algebra, (e.g. see [Web13]), we only work to the level of Frobenius algebra.

In Chapter 2 we review the diagonal LG $\mathscr{B}$-model construction to the level of a Frobenius algebra. In Chapter 3 we define the nonabelian maximal symmetry group and compute the complete symmetry group in certain cases. In Chapter 4 we develop the nonabelian construction to the level of a graded vector space. Due to the physical origin of many aspects of mirror symmetry, the vector space is called the *state space*. The vector space requires the definition of an appropriate group action and an additional grading, which we show to be well-defined. In Chapter 5 we introduce the pairing and an approach to the multiplication of the $\mathscr{B}$-model. This approach does not give a complete construction for all singularities and groups, and even when it does, it is not clear that the multiplication that is constructed is always associative. Nevertheless, it does give an associative product in many examples. In Chapter 6 we compute the $\mathscr{B}$-models, worked out to the level of Frobenius algebra, for the polynomials x^2y+y^3 and x^3y+y^4 with symmetry group $G_W^{sm} \cap SL(2,\mathbb{C})$ and the polynomial $x^3+y^3+z^3+w^2$ with symmetry group S_3.

Here we review briefly the $\mathscr{B}$-model construction for diagonal symmetry groups, as given in [Kra10].

The $\mathscr{B}$-model $\mathscr{B}_W^G$ is a graded $\mathbb{C}$-Frobenius algebra with a G-action, where G is a group. A $\mathbb{C}$-*Frobenius algebra* is an algebra A over $\mathbb{C}$ with a nondegenerate pairing that satisfies the Frobenius property. A pairing on A is a symmetric, bilinear, function $\langle \cdot, \cdot \rangle : A \times A \to \mathbb{C}$ and the Frobenius property is that $\langle ab, c\rangle = \langle a, bc\rangle$ for all $a, b, c \in A$. Our algebra is the G-invariant subspace of a direct sum of rings, one for each group element. (We define the group G and its action on the algebra later.) We write it

$$\mathscr{B}_W^G = \left(\bigoplus_{g \in G} \mathscr{H}_g \right)^G .$$

In the rest of the chapter, we review the construction of the group G, which we require to consist of diagonal matrices, the summands $\mathscr{H}_g$ of $\mathscr{B}_W^G$, the group action of G, the grading, the pairing, and the multiplication.

Weights and Group

The LG $\mathscr{B}$-model depends on a quasihomogeneous, nondegenerate polynomial $W \in \mathbb{C}[x_1, \ldots, x_n]$ and a choice G of symmetry group of W. A *symmetry* is an automorphism $g : \mathbb{C}^n \to \mathbb{C}^n$ such that $W \circ g = W$ when we consider W as a function on $\mathbb{C}^n$. The original construction requires *diagonal* symmetries, or symmetries with matrix representations that are diagonal matrices (see example 2.1). We denote the set of all diagonal symmetries of W by G_W^m, and note that this is a group.

The existence of an interesting group of symmetries follows since W is quasihomoegenous. Recall that being quasihomogeneous means there are rational numbers $q_1, \ldots, q_n$ such that $W(\lambda_1^q x_1, \ldots, \lambda_n^q x_n) = \lambda W$.

The $\mathscr{B}$-model construction requires symmetries with determinant 1. Thus, any group G in $G_W^m \cap SL(n, \mathbb{C})$ allows for a $\mathscr{B}$-model construction. Since any matrix of G_W^m is diagonal, a row vector containing the diagonal entries has all the essential information of the matrix. Furthermore, since each diagonal entry is a root of unity, we keep track of the entries by the fraction of $2\pi i$ in the exponent, modulo $\mathbb{Z}$. The entries of the row vectors in this format are the *phases* of g. This correspondence gives a group isomorphism from G_W^m to $\mathbb{Q}^n/\mathbb{Z}^n$. Note that since the group elements are all diagonal, they all commute; hence we write the group operation as addition.

Example 2.1. Let $W = P_8 = x^3 + y^3 + z^3$, with $a = 0$, and $G = G_W^m \cap SL$. Note that W has weights (1/3,1/3,1/3) since

$$(e^{1/3(2\pi i)}x)^3 + (e^{1/3(2\pi i)}y)^3 + (e^{1/3(2\pi i)}z)^3 = e^{2\pi i}(x^3 + y^3 + z^3) = W.$$

The group G has nine elements and is generated by

$$(0, 1/3, 2/3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{4\pi i/3} \end{pmatrix}, \quad (1/3, 0, 2/3) = \begin{pmatrix} e^{2\pi i/3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{4\pi i/3} \end{pmatrix}.$$

We write

$$\begin{aligned} G \;\; = \;\; & \{(0,0,0), (0,1/3,2/3), (0,2/3,1/3), \\ & (1/3,0,2/3), (2/3,0,1/3), (1/3,2/3,0), \\ & (2/3,1/3,0), (1/3,1/3,1/3), (2/3,2/3,2/3)\}. \end{aligned}$$

□

Determining $\mathscr{H}_g$

With the symmetry group defined, we can define the algebra $\mathscr{B}_W^G$. Recall that a $\mathbb{C}$-algebra is necessarily a $\mathbb{C}$-vector space. The construction of $\mathscr{B}_W^G$ begins with a vector space, on which we impose a multiplication and pairing.

The vector space is the sum of vector spaces (actually rings) $\mathscr{H}_g$, one for each element of the group. Each $\mathscr{H}_g$ is a *Milnor Rings.* The Milnor ring of a quasihomogeneous, nondegenerate polynomial W is the ring $\mathscr{Q}_W := \mathbb{C}[x_1, \ldots, x_n]/JW$, where $JW := \langle \frac{\partial W}{\partial x_1} \cdots \frac{\partial W}{\partial x_n} \rangle$ is called the *Jacobian Ideal.* That is, we mod out by the ideal generated by the partial derivatives of W.

Example 2.2. Let $W = x^2y + y^3$. The partial derivatives are $2xy$ and $x^2 + 3y^2$. A list of basis elements for $\mathscr{Q}_W$ looks like $\{1, y, y^2, x\}$, where we have written representatives of each equivalence class.

Now let $W = x^3y+y^4$. A list of basis elements for $\mathscr{Q}_W$ looks like $\{1, y, y^2, y^3, x, xy, xy^2, xy^3, x^2\}$.

If $W = P_8 = x^3 + y^3 + z^3$, here is a list of representatives of basis elements for $\mathscr{Q}_W$: $\{1, x, y, z, xy, xz, yz, xyz\}$. □

Now we can assemble the pieces to define $\mathscr{H}_g$. For each group element $g \in G$, there is a restriction of W that we use to make a Milnor ring associated to g. Let $V := \mathbb{C}^n$, then the *fixed locus* of g is the space $V^g = \{v \in \mathbb{C}^n : gv = v\}$. Since g is diagonal, there is a basis for V^g that is a subset of the standard basis. That is, each variable x_i is either fixed or not. The restriction $W|_{V^g}$ of W to V^g is obtained by setting x_i to 0 if it is not fixed by g. Then $\mathscr{H}_g$ is the quotient of $\mathbb{C}[x_1, \ldots, x_n]$ by the partials of $W|_{V^g}$ and the ideal generated by the non-fixed variables. Symbolically, this is

$$\mathscr{H}_g := \mathscr{Q}_{W|_{V^g}}.$$

Throughout this thesis, we denote elements of $\mathscr{H}_g$ by $\lfloor m, g \rceil$ where $m \in \mathscr{Q}_{W|_{V^g}}$. There is a basis of monomials for Milnor rings of quasihomogeneous, nondegenerate polynomials; these will be basis elements of $\mathscr{B}_W^G$

Example 2.3. For $W = x^3 + y^3 + z^3$ and $g = (0, 0, 0)$, all variables are fixed. Then $W|_{V^g} = W$ and $\mathscr{H}_g = \mathscr{Q}_W$. If $g = (0, 1/3, 2/3)$, then only x is fixed. $W|_{V^g} = x^3$ and $\mathscr{H}_g = \mathbb{C}[x, y, z]/(\langle y, z\rangle + \langle 3x^2\rangle)$, which has a basis $\{\lfloor 1, g\rceil, \lfloor x, g\rceil\}$.

$\square$

Group Action

There is a group action on $\mathscr{H}_g$ for each $g \in G$: If $m \in \mathscr{H}_g$ and $h \in G$,

$$m \cdot h := \det(h|_{V^g}) m \circ h.$$

Here, $h|_{V^g}$ is the restriction of h to V^g and $m \circ h$ is standard function composition. With this group action we can define $\mathscr{B}_W^G$ as the G-invariant subspace of $\bigoplus_{g\in G} \mathscr{H}_g$:

$$\mathscr{B}_W^G := \left(\bigoplus_{g\in G} \mathscr{H}_g \right)^G = \bigoplus_{g\in G} \mathscr{H}_g^G.$$

The second equality results from the diagonal action of G.

Example 2.4. Again, $W = P_8 = x^3 + y^3 + z^3$, with $a = 0$, and $G = G_W^m \cap SL$.

Consider $\mathscr{H}_{(0,1/3,2/3)} = \mathrm{Span}_{\mathbb{C}}\{1, x\}$. Note that $(0, 1/3, 2/3)$ restricts to the identity on this space. On the other hand, $(1/3, 0, 2/3) = \begin{pmatrix} e^{2\pi i/3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{4\pi i/3} \end{pmatrix}$ restricts to the map $c \mapsto e^{2\pi i/3} c$. Considering the action on x, we have $x \cdot (1/3, 0, 2/3) = \det(1/3, 0, 2/3)(e^{2\pi i/3} x) = e^{4\pi i/3} x$. Then x is not G-invariant.

In fact, the invariant basis elements are $\lfloor 1, (0, 0, 0) \rceil$, $\lfloor 1, (1/3, 1/3, 1/3) \rceil$, $\lfloor 1, (2/3, 2/3, 2/3) \rceil$, and $\lfloor xyz, (0, 0, 0) \rceil$.

Grading

A graded algebra is a direct sum of submodules that are indexed by a group such that when an element of a g-summand is multiplied by an element of an h-summand, the product is in the gh-summand. Our algebra, $\mathscr{B}_W^G$, is a Frobenius algebra with multiplication that respects multiple gradings. The definition introduces a grading by the symmetry group, which will be preserved by the multiplication. Additionally, we have two more gradings which the multiplication must preserve. For reference, we include the $\mathscr{A}$-model grading, which differs slightly from the $\mathscr{B}$-model grading.

First, we define the *weighted degree* of an element. If an element has a monomial representative $\prod_i x_i^{a_i}$, it has a well-defined weighted degree $\sum(a_i+1)q_i$. Say $\alpha \in \mathscr{H}_1 = \mathscr{Q}_W$, for example $\lfloor x, 1 \rceil$ with $W = x^3+y^3+z^3$. The weighted degree is $(1+1)1/3+(0+1)*1/3+(0+1)*1/3 = 4/3$. For an element in $\mathscr{H}_g$, the weighted degree uses the weights of $W|_{V^g}$. If $g = (0, 1/3, 2/3)$, then $W|_{V^g} = x^3$ and $\deg(\lfloor x, (0, 1/3, 2/3) \rceil) = (1 + 1) * 1/3 = 2/3$.

Under the weighted degree, each ring $\mathscr{H}_g$ decomposes into a sum over weights : $\mathscr{H}_g = \bigoplus_{q\in\mathbb{Q}} \mathscr{H}_{g,q}$. After a few more preliminary definitions, we can define the other gradings preserved by the multiplication. Let $h \in G$ and $\{q_1, \ldots, q_n\}$ be the weights of W. Let $\alpha \in \mathscr{H}_h$ have a well-defined weighted degree, which we'll denote $\deg \alpha$. Recall that the *age* of h is the sum of the phases, which are the fractions of $2\pi i$ in the exponent of the diagonal entries of h. If using the row vector notation for diagonal symmetry group elements, the phases are just the entries of the row vector. See example 2.5 below.

We have all the information to define the $\mathscr{B}$ bi-grading of $\alpha \in \mathscr{H}_g$, denoted with pluses and minuses $(\deg_+\lfloor \alpha, g \rceil, \deg_-\lfloor \alpha, g \rceil)_B$.

$$(\deg_+\lfloor \alpha, g \rceil, \deg_-\lfloor \alpha, g \rceil)_B := (\deg \alpha, \deg \alpha) + (\text{age } h, \text{age } h^{-1}) - (\sum_{i=1}^{n} q_i, \sum_{i=1}^{n} q_i)$$

It is important to note that while the weighted degree of $\lfloor m, g \rceil$ uses the weights of $W|_{V^g}$, the sum of the weights in the bi-grading is the sum of the weights of W.

Example 2.5. Again, $W = P_8 = x^3 + y^3 + z^3$, with $a = 0$, and $G = G_W^m \cap SL$. If $h = (0, 1/3, 2/3)$, then the age of h is $1/3 + 2/3 = 1$. Similarly, the age of $h^{-1} = (0, 2/3, 1/3)$ is $2/3 + 1/3 = 1$. The bi-degree of $\lfloor x, (0, 1/3, 2/3) \rceil = (2/3, 2/3) + (1, 1) - (1, 1) = (2/3, 2/3)$. $\square$

For reference we give the $\mathscr{A}$ grading of $\alpha \in \mathscr{H}_g$ when α has a well-defined weighted degree. We will need $N_h := \dim V^h$.

$$(\deg_+\lfloor\alpha, g\rceil, \deg_-\lfloor\alpha, g\rceil)_A := (\deg\alpha, N_h - \deg\alpha) + (\text{age } h, \text{age } h) - (\sum_{i=1}^{n} q_i, \sum_{i=1}^{n} q_i)$$

<u>Pairing</u>

The Hessian of a polynomial is the determinant of the second derivative matrix $(\frac{\partial^2 W}{\partial x_i \partial x_j})$. Let $\mu_g := \dim \mathscr{H}_g$. For $p, q \in \mathscr{H}_g = \mathscr{Q}_{W|_{V^g}}$, the *residue pairing* $\langle p, q\rangle$ is defined implicitly as the solution of the equation

$$pq = \frac{\langle p, q\rangle}{\mu_g}\text{Hess } W|_{V^g} + \text{lower order terms}$$

. We note that $W|_{V^g}$ is obtained from W by setting to 0 all variables not fixed by g. Thus it is sensible to take derivatives and discuss the Hessian of $W|_{V^g}$.

Since Fix $h =$ Fix h^{-1}, we have $\mathscr{H}_h \cong \mathscr{H}_{h^{-1}}$, and the residue pairing on $\mathscr{Q}_{W|_{\text{Fix } h}}$ induces a pairing

$$\mathscr{H}_h \otimes \mathscr{H}_{h^{-1}} \to \mathbb{C}.$$

The pairing on $\mathscr{H}$ is the direct sum of these pairings. Fixing a basis for $\mathscr{H}$, we denote the pairing by a matrix $\eta_{\alpha,\beta} = \langle\alpha, \beta\rangle$.

Example 2.6. Again, $W = P_8 = x^3 + y^3 + z^3$, with $a = 0$, and $G = G_W^m \cap SL$. Let x, yz be elements of $\mathscr{H}_1 = \mathscr{Q}_W$.

The Hessian of $W = x^3 + y^3 + z^3$ is $216xyz$ and the dimension of $\mathscr{Q}_W = 8$. Since x and yz multiply to xyz, the equation $\langle x, yz\rangle * 216xyz/8 + ... = x(yz)$ indicates that the pairing $\langle x, yz\rangle = 1/27$. The pairing matrix of $\mathscr{H}_1 = \mathscr{Q}_W$ is

Alternatively, this could be written $\langle x^a y^b z^c, x^d y^e z^f\rangle = \begin{cases} 1/27 & a + d = 1 = b + e = c + f \\ 0 & \text{otherwise} \end{cases}$

We use both the matrix and the function notation, depending on the size of $\mathscr{H}_g$.

<u>Multiplication</u>

$\mathscr{H}_1$	1	x	y	z	xy	xz	yz	xyz
1								$1/27$
x							$1/27$	
y						$1/27$		
z					$1/27$			
xy				$1/27$				
xz			$1/27$					
yz		$1/27$						
xyz	$1/27$							

Table 2.1: The pairing matrix of $\mathscr{Q}_W = \mathscr{H}_1$ for $W = P_8 = x^3 + y^3 + z^3$.

With the grading and pairing established and the space $\mathscr{B}_W^G$ understood, we can introduce the $\mathscr{B}$-model multiplication. It is defined on basis elements and extended linearly.

Let $\lfloor m, g\rceil$ denote the element m of H_g. Let F_g denote the indices of the variables fixed by g. The product of $\lfloor m, g\rceil$ and $\lfloor n, h\rceil$ is given by

$$\lfloor m, g\rceil \star \lfloor n, h\rceil := \lfloor \gamma_{g,h} mn, g+h\rceil,$$

where $\gamma_{g,h}$ satisfies the equation

$$\gamma_{g,h} \frac{\mathrm{Hess} W|_{V^g \cap V^h}}{\dim \mathscr{Q}_{W|_{V^g\cap V^h}}} = \begin{cases} \dfrac{\mathrm{Hess}\ W_{V^{gh}}}{\dim \mathscr{Q}_{W|_{V^{gh}}}} & \text{if } F_g \cup F_h \cup F_{gh} = \{1, \dots, n\} \\ 0 & \text{else} \end{cases}.$$

Example 2.7.

In [FJJS12], an alternative expression of the multiplication was explained and it was shown to be equivalent to our definition above. It involves the following diagram

Here $\mathscr{H}_{g,h}$ is $\mathscr{Q}_{W|_{V^g\cap V^h}}$ and $\widehat{\mathscr{H}_{g,h}}$ and $\widehat{\mathscr{H}_{gh}}$ are the dual vector spaces to $\mathscr{H}_{g,h}$ and $\mathscr{H}_{gh}$, respectively. The maps from $\mathscr{H}_g$, $\mathscr{H}_h$, and $\mathscr{H}_{gh}$ to $\mathscr{H}_{g,h}$ are all projections. The map from $\widehat{\mathscr{H}_{g,h}}$ to $\widehat{\mathscr{H}_{gh}}$ is the injective adjoint to the projection above it.

To multiply $\lfloor m, g\rceil$ and $\lfloor n, h\rceil$, we project into $\mathscr{H}_{g,h}$, multiply by a special factor $\epsilon_{g,h}$,

$$
\begin{tikzcd}
\mathscr{H}_g \arrow[dr] & & \\
& \mathscr{H}_{g,h} \arrow[dd, "\eta^\flat"] & \mathscr{H}_{gh} \arrow[l, two heads] \\
\mathscr{H}_h \arrow[ur] & & \\
& \widehat{\mathscr{H}_{g,h}} \arrow[r, tail] & \widehat{\mathscr{H}_{gh}} \arrow[uu, "\eta^\sharp"']
\end{tikzcd}
$$

Figure 2.1: The multiplication diagram

then trace this through $\eta^\flat$, the adjoint, and $\eta^\sharp$. In the diagonal case, the $\epsilon_{g,h}$ factor is

$$\epsilon_{g,h} = \begin{cases} 1 & \text{if } F_g \cup F_h \cup F_{gh} = \{1, \ldots, n\} \\ 0 & \text{else} \end{cases}.$$

This gives the piecewise nature of $\gamma_{g,h}$; the rest of the coefficient comes from tracing through the diagram.

This diagram presentation of the product is the mechanism we use to generalize the product in Chapter 5.

In the diagonal case, this construction is a graded Frobenius algebra with a group action. In the following chapters, we generalize this construction to nondiagonal symmetry groups. Some definitions apply readily and some require more thought.

Chapter 3. Groups

As discussed in 2, the LG $\mathscr{B}$- model depends on a quasihomogeneous, nondegenerate polynomial $W \in \mathbb{C}[x_1, \ldots, x_n]$ and the choice G of symmetry group of W. In this chapter we discuss the conditions we impose on the group of symmetries when we don't require all symmetries to be diagonal. By diagonal or off-diagonal, we are considering the matrix representation of a symmetry with respect to the standard basis on $\mathbb{C}^n$. We often use the words nondiagonal or off-diagonal to describe symmetries of W that may have nonzero elements off the diagonal. After defining the maximal group of nondiagonal symmetries, we we determine explicit matrix representations of admissible symmetries for certain polynomials. Throughout, we compare conditions in the diagonal case with the nondiagonal case.

Example 3.1. $P_8: \ x^3 + y^3 + z^3 + axyz$ admits diagonal symmetries such as $\begin{pmatrix} e^{(2\pi i)/3} & 0 & 0 \\ 0 & e^{(2\pi i)/3} & 0 \\ 0 & 0 & e^{(2\pi i)/3} \end{pmatrix}$. It also permits off-diagonal symmetries such as $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which simply permutes the first two standard basis elements of $\mathbb{C}^3$. We identify this symmetry with the permuation (12). We see that P_8 admits any permutation from the symmetric group, so S_3 is a subset of the symmetries of P_8. □

Recall that the set G_W^m of maximal diagonal symmetries of W consists of all diagonal symmetries. When we allow for nondiagonal matrices, we have to expand our definition of the symmetry group. To understand the definition of the maximal group of symmetries of W, we introduce what is known as the $\mathbb{C}^\times$ operator, denoted here as f_λ. Since W is a quasihomogoneous polynomial, it has a weight system $q_1, \ldots, q_m$ where each q_i is a rational number. This means that $W(\lambda^{q_1}c_1, \lambda^{q_2}c_2, \ldots, \lambda^{q_m}c_m) = \lambda W(c_1, \ldots, c_m)$ for any $\lambda \neq 0$ in $\mathbb{C}$. If $\{e_1, \ldots e_m\}$ is the standard basis for $\mathbb{C}^m$, this is the same as saying $W(\sum_{i=1}^m \lambda^{q_i} c_i e_i) = \lambda W(\sum_{i=1}^m c_i e_i)$. This leads to the definition of the $\mathbb{C}^\times$ operator $f_\lambda : \mathbb{C}^m \to \mathbb{C}^m$ by $f_\lambda(e_i) = \lambda^{q_i} e_i$.

We require each symmetry in the symmetry group to commute with the $\mathbb{C}^\times$ operator. That is, for g a symmetry of W, the equation $f_\lambda g = g f_\lambda$ holds for all $\lambda \in \mathbb{C}^\times$. This commutativity of operators occurs in the diagonal case since f_λ is also a diagonal linear automorphism of $\mathbb{C}^n$, so clearly commutes with other diagonal linear automorphisms of $\mathbb{C}^n$. When we allow for nondiagonal automorphisms, the commutativity is not so obvious, so we impose this condition on our maximal symmetry group. Since the maximal diagonal symmetry group is called the maximal symmetry group, we call the larger group of symmetries the “super-maximal” symmetry group, or G_W^{sm}.

Definition 3.2. Let W be a quasihomogeneous, nondegenerate polynomial with weights $< 1/2$, let $q_1, \ldots, q_n$ be the weights, and let f_λ be the $\mathbb{C}^\times$ operator. G_W^{sm} is the set of all $g \in \mathrm{Aut}\ C^n$ such that $W \circ g = W$ and $gf_\lambda = f_\lambda g$.

Example 3.3. P_8 has unique weights, so it has a unique $\mathbb{C}^\times$ operator $\begin{pmatrix} \lambda^{1/3} & 0 & 0 \\ 0 & \lambda^{1/3} & 0 \\ 0 & 0 & \lambda^{1/3} \end{pmatrix}$.

Let (12) be the symmetry $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. If we compute $f_\lambda(12)$ and $(12)f_\lambda$ (rather, the product of their matrices) we get $\begin{pmatrix} 0 & \lambda^{1/3} & 0 \\ \lambda^{1/3} & 0 & 0 \\ 0 & 0 & \lambda^{1/3} \end{pmatrix}$ for both. Since P_8 is homogeneous, λ is a scalar matrix and so lies in the center of $M_3(C)$. Thus every automorphism of P_8 lies in G_W^{sm}.

3.1 When does a singularity admit a nonabelian symmetry group?

We now consider which polynomials admit a nonabelian symmetry group; that is, when G_W^m is a proper subset of G_W^{sm}. We do not yet know of a sufficient condition for a nonabelian symmetry group, but we can establish a necessary condition for nonabelian symmetries. Recall each symmetry in the symmetry group must commute with the $\mathbb{C}^\times$ operator. That is, for $g \in G_W^{sm}$, $f_\lambda g = gf_\lambda$ for all $\lambda \in \mathbb{C}^\times$.

Proposition 3.4. *If (g_{ij}) is the matrix representation of $g : \mathbb{C}^n \to \mathbb{C}^n$, then $f_\lambda g = gf_\lambda$ if and only if $g_{ij} \neq 0$ implies that $q_i = q_j$.*

Proof. Consider the actions of $f_\lambda g$ and gf_λ on each basis element e_i, $i = 1, \ldots, m$, of $\mathbb{C}^n$.

We have

$$f_\lambda g(e_i) = f_\lambda(\sum_{j=1}^{n} g_{ij}e_j) = \sum_{j=1}^{n} g_{ij}\lambda^{q_j}e_j.$$

Also,

$$gf_\lambda(e_i) = g(\lambda^{q_i}e_i) = \lambda^{q_i}\sum_{j=1}^{n} g_{ij}e_j.$$

Note that if $f_\lambda g = gf_\lambda$, then

$$\sum_{j=1}^{n} g_{ij}\lambda^{q_j}e_j = \sum_{j=1}^{n} \lambda^{q_i}g_{ij}e_j$$

for all i. Since the e_i are a basis for $\mathbb{C}^n$, they are linearly independent and the equality above indicates that $g_{ij}\lambda^{q_j} = g_{ij}\lambda^{q_i}$. So $g_{ik} \neq 0$ implies that $\lambda^{q_k} = \lambda^{q_i}$ for all λ, meaning $q_k = q_i$.

Conversely, if $g_{ij} \neq 0$ implies $q_i = q_j$, then $f_\lambda g(e_i) = \sum_{j=1}^{n} g_{ij}\lambda^{q_j}e_j = \lambda^{q_i}\sum_{j,\ g_{ij}\neq 0} g_{ij}e_j = gf_\lambda e_i$.

□

This shows that if we require G_W^{sm} to contain only automorphisms of $\mathbb{C}^n$ that commute with the $\mathbb{C}^\times$ action, then the symmetries only act among subspaces having the same weights. If we decompose $\mathbb{C}^n$ by weights, then the matrix representations of symmetries are block diagonal. Also, if all the weights q_i are distinct, then $G_W^{sm} = G_W^m$ and the polynomial doesn't admit a nonabelian symmetry group. This suggests that homogeneous polynomials are a natural place to begin looking for nonabelian symmetry groups, since all the weights are the same.

In the quasihomogeneous, nondegenerate cases considered so far, all symmetries commute with the $\mathbb{C}^\times$ action. We conjecture that there are polynomials with symmetries that don't commute with the $\mathbb{C}^\times$ action, but these are in a small minority among nondegenerate, quasihomogeneous polynomials.

3.2 What can we say about the "supermax" symmetry groups (G^{sm}) of atomic type polynomials?

Recall from Chapter 1 that many results are simpler in the case of certain nice polynomials, which we called atomic types. In this section we present some results on the symmetry groups of these polynomials. Because of Proposition 3.4, interesting symmetries occur when variables have the same weight. Hence we only consider homogeneous atomic type polynomials.

We'll start with Fermats. A homogeneous Fermat has the form $\sum_{i=1}^{n} x_i^a$. It's easy to see that permutation matrices with a-th roots of unity in the nonzero spots will be symmetries of a homogeneous Fermat. We write these matrices as $S_n \cdot D$ where S_n is an $n \times n$ permutation matrix and D_a is an $n \times n$ diagonal matrix with a-th roots of unity on the diagonals. In writing out the general equations to solve for symmetries of Fermats, there are many equations that equal 0. This suggests that the matrices described are the only symmetries of sums of Fermats. This isn't proved in general, but we present a proof of the two-variable case here. Note that the theorem is for Fermats with degree greater than 3. The polynomial $x^2 + y^2$ is a special case - it actually has an infinite symmetry group that we don't discuss in this thesis.

Theorem 3.5. *Let $W = x^n + y^n$ be a sum of Fermats. If $n \geq 3$, then G_W^{sm} consists of $S_2 \cdot D_n$; that is, G^{sm} consists of matrices which are the product of an 2×2 permutation matrix and a diagonal matrix of the form $\zeta_i \delta_{ij}$ where δ_{ij} is the Kronecker delta and ζ_i is an a-th root of unity.*

Proof. Let $W = x^n + y^n$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, We want $W \circ g = W$, which means $(ax+by)^n + (cx+dy)^n = x^n + y^n$. If we expand and equate the coefficients of the $x^n, x^{n-1}y, x^{n-2}y^2$, and

y^n terms, we get the following equations:

$$a^n + c^n = 1$$
$$n(a^{n-1}b + c^{n-1}d) = 0$$
$$n(a^{n-2}b^2 + c^{n-2}d^2) = 0$$
$$b^n + d^n = 1$$

We see that if $a = 0$, then $c^n = 1$, $d = 0$, and $b^n = 1$. Similarly, if $d = 0$ then $b^n = 1$, $a = 0$, and $c^n = 1$. Similarly, $b = 0$ iff $c = 0$ iff $a^n = d^n = 1$.

Now suppose $a, b, c, d \neq 0$. Multiply the second equation by b, the third by a, and subtract.

$$a^{n-2}b + bc^{n-1}d = 0$$
$$-(a^{n-1}b^2 + ac^{n-2}d^2) = 0$$
$$c^{n-2}d(bc - ad) = 0$$

So $bc = ad$. From the original set of equations, take the second and multiply by a: $a(a^{n-1}b + c^{n-1}d = 0)$ gives $a^n b + adc^{n-1} = 0$, which gives $a^n b + bc^n = 0$, or $b(a^n + c^n) = 0$, or $b = 0$. This is a contradiction, so we conclude that $a = d = 0$ or $c = b = 0$.

□

We move to the next atomic type, loop polynomials. Homogeneous loop polynomials have the form $x_1^a x_2 + x_2^a x_3 + \ldots + x_n^a x_1$. As with sums of Fermats, we have results for two-variable loops. The loops $x^2y + y^2x$ and $x^3y + y^3x$ don't satisfy the pattern in our theorem since they have small homogeneous degree. However, the groups are easily computable with most mathematical software and aren't presented in this thesis. Here we present a result on the symmetries of two-variable loops of degree greater than 5.

Theorem 3.6. *Let $W = x^n y + xy^n$ where $n = 4$ or $n \geq 6$. The only symmetries are of the* *form* $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ *or* $\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$, *where* $\alpha^{n^2-1} = 1$ *and* $\beta = \alpha^{-n}$. *We call the* $\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$ *matrices* off-diagonal. *The order of the group is* $2(n^2 - 1)$. *If* $W = x^5y + xy^5$, *there are more than the diagonal or off-diagonal symmetries, and* $|G^{sm}| = 144$.

Proof. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We require that $W \circ g = W$, or

$$(ax + by)^n(cx + dy) + (ax + by)(cx + dy)^n = x^ny + y^nx$$

. Expanding, we get

$$\left(\sum_{j=0}^{n} \begin{pmatrix} n \\ j \end{pmatrix} (ax)^j(by)^{n-j}\right)(cx + dy) + (ax + by)\left(\sum_{j=0}^{n} \begin{pmatrix} n \\ j \end{pmatrix} (cx)^j(dy)^{n-j}\right) = x^ny + xy^n.$$

We distribute and relate coefficients to determine g. First, we'll examine the coefficients of $x^{n+1}, x^ny, x^{n-1}y^2, x^2y^{n-1}, xy^n$, and y^{n+1}:

$$a^nc + ac^n = 0 \tag{3.1}$$

$$na^{n-1}bc + a^nd + nac^{n-1}d + bc^n = 1 \tag{3.2}$$

$$\frac{n(n-1)}{2}a^{n-2}b^2c + na^{n-1}bd + \frac{n(n-1)}{2}ac^{n-2}d^2 + nbc^{n-1}d = 0 \tag{3.3}$$

$$nab^{n-1}c + \frac{n(n-1)}{2}a^2b^{n-2}d + nacd^{n-1} + \frac{n(n-1)}{2}bc^2d^{n-2} = 0 \tag{3.4}$$

$$b^nc + nab^{n-1}d + ad^n + nbcd^{n-1} = 1 \tag{3.5}$$

$$b^nd + bd^n = 0 \tag{3.6}$$

As before, we get that $a = 0$ if and only if $d = 0$, and in this case, $bc^n = 1$ and $b^nc = 1$, or $b^{n^2-1} = 1$ and $c = b^{-n}$. Similarly, $c = 0$ if and only if $b = 0$, and $a^{n^2-1} = 1$, $d = a^{-n}$.

Now suppose that all coefficients are not 0. Eqns. 3.1 and 3.6 give us $ac(a^{n-1} + c^{n-1}) = 0$

and $bd(b^{n-1} + d^{n-1}) = 0$. We conclude that $a = \sqrt[n-1]{-1}c$ and $b = (-1)^{1/(n-1)}d$. Let w_n be an $(n-1)$th root of -1 and $a = w_n c$. Then $b = w_n^{\alpha} d$ for some $\alpha \in \mathbb{Z}$. To determine α, substitute in Eqn. 3.3 to get

$$\begin{aligned}
\frac{n(n-1)}{2}(w_n c)^{n-2}(w_n^{\alpha} d)^2 c + n(w_n c)^{n-1}(w_n^{\alpha} d)d + \frac{n(n-1)}{2}(w_n c)c^{n-2}d^2 + n(w_n^{\alpha} d)c^{n-1}d &= 0 \\
\frac{n(n-1)}{2} w_n^{n-2+2\alpha} c^{n-1} d^2 + n - w_n^{\alpha} c^{n-1} d^2 + \frac{n(n-1)}{2} w_n c^{n-1} d^2 + n w_n^{\alpha} c^{n-1} d^2 &= 0 \\
\frac{n(n-1)}{2} c^{n-1} d^2 w_n (w_n^{n-3+2\alpha} + 1) &= 0
\end{aligned}$$

This implies that $w_n^{n-3+2\alpha} = -1$, meaning $n - 3 + 2\alpha = (2k+1)(n-1)$ is an odd multiple of $n-1$. This gives $n - 3 + 2\alpha = 2k(n-1) + n - 1$, or $\alpha = k(n-1) + 1$.

Substitute in Eqn. 3.2 to get

$$\begin{aligned}
n(w_n c)^{n-1}(w_n^{\alpha} d)c + (w_n c)^n d + n(w_n c)c^{n-1}d + (w_n^{\alpha} d)c^n &= 1 \\
c^n d(-n w_n^{\alpha} - w_n + n w_n + w_n^{\alpha}) &= 1
\end{aligned}$$

The only problems arise when $-n w_n^{\alpha} - w_n + n w_n + w_n^{\alpha} = 0$, meaning (recall that $|w_n^{\alpha}| = 1$), $w_n^{\alpha} = w_n$, or $w_n^{\alpha - 1} = 1$. This occurs precisely when $\alpha - 1 = 2l(n-1)$ is an even multiple of $n-1$. So we require that $\alpha = (2m+1)(n-1) + 1 = 2m(n-1) + n - 1 + 1$. Then $w_n^{\alpha} = w_n^{\alpha} = (w_n^{n-1})^{2m} w_n^{n-1} w_n = -w_n$. So $b = -w_n d$.

We discuss the $n = 4$ case separately. For $n \geq 5$, we use the equation for the coefficients of $x^{n-2}y^3$,

$$0 = \frac{n(n-1)(n-2)}{6}a^{n-3}b^3c + \frac{n(n-1)}{2}a^{n-2}b^2d + \frac{n(n-1)(n-2)}{6}ac^{n-3}d^3 + \frac{n(n-1)}{2}bc^{n-2}d^2$$
$$0 = \frac{n(n-1)(n-2)}{6}(w_nc)^{n-3}(-w_nd)^3c + \frac{n(n-1)}{2}(w_nc)^{n-2}(-w_nd)^2d$$
$$\ldots + \frac{n(n-1)(n-2)}{6}(w_nc)c^{n-3}d^3 + \frac{n(n-1)}{2}(-w_nd)c^{n-2}d^2$$
$$0 = c^{n-2}d^3w_n\left(\frac{n(n-1)(n-2)}{6}(--1+1) + \frac{n(n-1)}{2}(-1-1)\right)$$
$$0 = c^{n-2}d^3w_nn(n-1)\left(\frac{n-2}{3}-1\right).$$

This happens only when $n = 1$ or $n = 5$. Hence for $n \geq 6$, the only symmetries are the diagonal or off-diagonal matrices.

To show that for $n = 4$ the only symmetries are diagonal or off-diagonal, and to investigate the symmetries for $n = 5$, we continue our assumption that a, b, c, and d are all nonzero and substitute $a = w_nc$ and $b = -w_nd$ into Eqns. 3.2 and 3.5. We get

$$\begin{aligned} n(w_nc)^{n-1}(-w_nd)c + (w_nc)^nd + n(w_nc)c^{n-1}d + (-w_nd)c^n &= 1, \\ (-w_nd)^nc + nw_nc(-w_nd)^{n-1}d + w_ncd^n + n(-w_nd)cd^{n-1} &= 1. \end{aligned}$$

These simplify to

$$\begin{aligned} w_nc^nd(2n-2) &= 1, \\ w_ncd^n((-1)^{n+1} + n(-1)^n + 1 - n) &= 1. \end{aligned}$$

If n is even, then we have $((-1)^{n+1} + n(-1)^n + 1 - n) = 0$ so $0 = 1$, a contradiction. We conclude that $x^4y + xy^4$ has only diagonal and off-diagonal symmetries. As a side note, every group presented so far has order $|G_L^{sm}| = 2(n^2-1)$, since there are $n^2 - 1$ choices of entries for the diagonal matrices $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ and $n^2 - 1$ choices of entries for the off-diagonal matrices $\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$.

If $n = 5$, then the two equations we get for c and d become

$$\begin{aligned} w_5 c^5 d(8) &= 1, \\ w_5 c d^5(-8) &= 1. \end{aligned}$$

Solving, we get

$$\begin{aligned} c &= \zeta_{24} \frac{1}{\sqrt[6]{8}} \\ d &= \frac{1}{w_5 c^n 8} \end{aligned}$$

where ζ_{24} is a 24th root of unity. There are 4 choices for $w_5 = \sqrt[4]{-1}$ and 24 choices for ζ_{24}, so we obtain 96 symmetries in addition to the 48 diagonal/off-diagonal symmetries. Then $|G_{L55}^{sm}| = 144$.

□

Finally, we offer a remark on chain polynomials. Homogeneous chains have the form $x_1^a x_2 + \ldots + x_{n-1}^a x_n + x_n^{a+1}$. If W is a chain, G_W^{sm} is usually diagonal since the equations require a-th and $a+1$-th roots of unity to be compatible in some sense. Some instances of chain polynomials with nondiagonal symmetries are presented in Chapter 6, but we don't yet have results for general chain polynomials.

3.3 Admissible Groups for $\mathscr{B}$- models

There is one last condition on the symmetry group to discuss before we move on in our $\mathscr{B}$-model construction. For the $\mathscr{B}$-model construction, we follow the example of Krawitz

[Kra10] and require G to be in SL, meaning all symmetries have determinant 1. This is necessary for the identity of the $\mathscr{B}$-model multiplication to be G-invariant (once we define the action of G on $\mathscr{B}_W^G$!), so that $\mathscr{B}_W^G$ has an identity and actually is a $\mathbb{C}$-algebra. We have $1 \in G \subset G_W^{sm} \cap \mathrm{SL}(n, \mathbb{C}) \subset G_W^{sm}$ for any $\mathscr{B}$- admissible group G.

Example 3.7. For the chain polynomial $x^2y + y^3$, the group $G_W^{sm} \cap SL$ has three elements:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; -\frac{1}{2}\begin{pmatrix} 1 & 3i \\ i & 1 \end{pmatrix}, \; \frac{1}{2}\begin{pmatrix} -1 & 3i \\ i & -1 \end{pmatrix},$$

so $G_W^{sm} \cap SL \cong \mathbb{Z}_3$. This is still abelian, so not ultimately fulfilling our goal of a nonabelian LG construction. On the other hand, this group is small and nondiagonal, so it will help us develop our generalization.

End Example

For reference, we mention the condition on admissible $\mathscr{A}$-model symmetry groups here. FJRW theory, which provides the $\mathscr{A}$-model construction, requires the group G to contain the *exponential grading operator* j. This is the (diagonal) linear map on $\mathbb{C}^n$ defined by sending $(c_1, \dots, c_n)$ to $(c_1 e^{2\pi i q_1}, \dots, c_n e^{2\pi i q_n})$.

Chapter 4. State Space

We are now in position to generalize the construction of $\mathscr{B}_W^G$ from the diagonal case to allowing for symmetries that aren't necessarily diagonal. In this chapter, we construct $\mathscr{B}_W^G$ when G is a nondiagonal symmetry group and demonstrate that it is a graded vector space.

Recall from Chapter 2 that the construction of the algebra $\mathscr{B}_W^G$ involves a sum of Milnor rings, a group action on the sum, taking the group invariant subspace, and establishing a bi-grading. In this chapter, we generalize this construction and present some alternate expressions of the state space. We use the term *State Space* to describe the underlying set of the construction; this is a result of the physical origin of the construction. The $\mathscr{B}$-

model construction was originally proposed by the high-energy physics community [BH92, IV90]. Following Krawitz in [Kra10], the sum of rings before taking invariants is called the *unprojected state space*, and written $\mathscr{H}_W$.

Remark. For convenience, when the indeterminates of a polynomial ring are clear, we will use a bolded notation. For example, $\mathbb{C}[x_1, \dots, x_n]$ will be written $\mathbb{C}[\mathbf{x}]$.

4.0.1 Detour.

We make here a detour of several pages to discuss an important technicality and some notation used often in our construction.

Recall that the symmetry group G_W^{sm} (we will often suppress the W and sm when the context is clear) is a subset of automorphisms on $V := \mathbb{C}^n$. If we choose a basis $\{e_1, \dots, e_n\}$ for $V = \mathbb{C}^n$, we can construct the canonical dual basis $\{x_1, \dots, x_n\}$ to the dual space $\widehat{V} := \widehat{\mathbb{C}^n}$. An automorphism $g \in G_W^{sm}$ induces an automorphism $\widehat{g}$ on the dual space $\widehat{\mathbb{C}}$ in the usual way, namely $\widehat{g}(f)\ (c_1, \dots, c_n) := f(g(c_1, \dots, c_n))$. This can be interpreted as an action on the vector space $P_1 := \{\textit{polynomials of degree 1}\}$, which extends to an action on the finitely generated $\mathbb{C}$-algebra $\mathbb{C}[\widehat{V}] := \mathbb{C}[x_1, \dots, x_n]$ by its universal mapping property.

We will try to distinguish between g the automorphism of W and $[g]$ its matrix representation. The symbol $\widehat{g}$ will denote the function on the dual space and on $\mathbb{C}[\widehat{V}]$ induced by g, and $[\widehat{g}]$ will denote the matrix representing the linear transformation of $\widehat{g}$ restricted to P_1.

Here is a table summarizing the spaces we work with, some correspondences we will often use without stating explicitly:

space	label	basis notation	algebraic correspondent
$\mathbb{C}^n$	V	$\{e_i\}$	
$\widehat{\mathbb{C}^n}$	$\widehat{V}$	$\{x_i\}$	P_1
functions on $\mathbb{C}^n$	$\mathbb{C}[\widehat{V}]$	$\{x_i\}$	$\mathbb{C}[x_1, \dots, x_n]$

Table 4.1: Notation we use commonly to keep track of the vector space and its dual.

It is useful here to demonstrate a few properties of a dual operator:

Proposition 4.1. *Let V, W be finite-dimensional vector spaces with bases $\{e_i\}$, $\{f_i\}$, respectively, and let $\widehat{V}, \widehat{W}$ be the duals, with canonical bases $\{x_i\}$, $\{y_i\}$, respectively. Let*

$T : V \to W$ *be a linear map defined by* $T(e_i) = t_{ji} f_j$.

(i) (the dual is the transpose) $\widehat{T} : \widehat{W} \to \widehat{V}$ *is defined by* $\widehat{T}(y_i) = t_{ij} x_j$ [1]

(ii) If T *is injective,* $\widehat{T}$ *is surjective. If* T *is surjective,* $\widehat{T}$ *is injective.*

(iii) $\det \widehat{g} = \det g$

(iv) Suppose $V = W$ *and* T *is diagonalizable. (That is,* $T = PDP^{-1}$ *where* D *is a diagonal linear transformation that scales by the eigenvalues* λ_i *of* T*, and the corresponding eigenvectors are of the form* $p_{ji} x_j$*.) Then the eigenvectors of* $\hat{T}$ *are of the form* $p_{ij}^{-1} x_j$*, with eigenvalues* λ_i *corresponding to those of* $p_{ji} x_i$.

Proof. We present some aspects of the proof, because they indicate notationally how we will often work with things in V versus things in $\mathbb{C}[\widehat{V}]$. More information can be found a linear algebra, basic abstract algebra, or functional analysis text.

(i) This comes by straightforward calculation on basis elements:

$$\begin{aligned} \widehat{T}(y_i)(e_j) &= y_i(Te_k) \\ &= y_i(t_{jk} f_j) \\ &= t_{jk} y_i(f_j) \\ &= t_{jk} \delta_{ij} \\ &= t_{ik} \\ &= t_{ij} x_k(e_j). \end{aligned}$$

Since $\widehat{T}(y_i)$ and $t_{ij} x_k$ agree on all basis elements, they are the same.

$\square$

[1] We will often employ Einstein summation notation, meaning we suppress writing $\Sigma_{i=1}^{n}$ and rely on repeated indices to indicate summation. Thus, $m_{ij} e_j = \sum_{i=1}^{n} m_{ij} e_j$. Eigenvalues require care but we will point this out when necessary. We use the convention $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

Remark. We make a note on notation here. Recall from Chapter 2 that we used the symbol $\lfloor m, g \rceil$ to denote an element of $\mathscr{H}_g$, where $g \in G_W^m$ and $m \in \mathscr{Q}_{W|_{V^g}}$. We haven't verified that this definition is sensible when g is a nondiagonal symmetry of W, but we shortly will. We will show that it is equivalent to discuss equivalence classes modulo ideals of $\mathbb{C}[\mathbf{x}]$, and we present the corresponding notation here. If I and J are ideals of $\mathbb{C}[\mathbf{x}]$ and m is an element of $\mathbb{C}[\mathbf{x}]$, we use $\lfloor m, I \rceil$ to denote the residue of m in $\mathbb{C}[\mathbf{x}]$ modulo I. Similarly, we use $\lfloor I, J \rceil$ to denote the ideal in $\mathbb{C}[\mathbf{x}]$ of residues of elements of I modulo J.

We use square brackets for matrix representations of linear transformations. For example, if T is a linear transformation on a space V with respect to bases E and B, then $[T]_E^B$ will denote the matrix representation of T. Often the bases are understood, so we just write $[T]$. This will be useful when discussing Jacobian ideals JW using the chain rule of differentiation.

4.1 $\mathscr{H}_g$

Now we resume the construction of $\mathscr{B}_W^G$; or rather, the unprojected state space $\bigoplus_{g \in G} \mathscr{H}_g$. In the diagonal case, $\mathscr{B}_W^G$ is the G-invariant subspace of $\bigoplus_{g \in G} \mathscr{H}_g$, where $\mathscr{H}_G$ is the Milnor ring of $W|_{V^g}$. We keep that same definition, but in this section we establish what $\mathscr{Q}_{W|_{V^g}}$ means when g is not necessarily diagonal.

First, we establish what $W|_{V^g}$ is when g is not diagonal. If $V := \mathbb{C}^n$, then the *fixed-point locus* of a group element g is the space $V^g := \{v \in V : \ gv = v\}$. Note that V^g is the eigenspace of g corresponding to the eigenvalue 1. Then $W|_{V^g}$ as a function is the function W restricted to V^g. To express $W|_{V^g}$ as a polynomial, we have to proceed more carefully than in the case where g is a diagonal symmetry where we merely set the unfixed variables of W to 0. This becomes a question of coordinates and a choice of basis for V^g. To that end, let $\{b_1, \ldots, b_k\}$ be a basis for V^g and $\{s_1, \ldots, s_k\}$ be the canonical dual basis to $\{b_1, \ldots, b_n\}$. The inclusion map $\iota : V^g \hookrightarrow V$ has a surjective adjoint $\widehat{\iota} : \widehat{V} \to \widehat{V^g}$ on the dual spaces. Extending this map to the polynomial ring $\mathbb{C}[s_1, \ldots, s_k] = \mathbb{C}[\widehat{V^g}]$, we can express $W|_{V^g}$ as $\widehat{\iota}(W) = W \circ \iota$.

Proposition 4.2. *$W|_{\text{fix } g}$ is a quasihomogeneous, nondegenerate polynomial in $\mathbb{C}[s_1, \ldots, s_k]$.*

Proof. First, we show that $W|_{V^g}$ is nondegenerate. Recall that this requires $W|_{V^g}$ to have an isolated singularity at the origin. We use the chain rule to study the derivative matrix of $W|_{V^g}$. If $f : \mathbb{C}^m \to \mathbb{C}^n$ is differentiable, let Df be the derivative matrix of f, so $(Df)_{ij} := \frac{\partial f_i}{\partial x_j}$. The map Df gives is a linear function from $\mathbb{C}^m$ to $\mathbb{C}^n$ at every point of $\mathbb{C}^n$. Now, since $W|_{V^g} = W \circ \iota$, we have $DW|_{V^g} = D(W \circ \iota) = (DW \circ \iota)D\iota$. This is the product of a row vector $DW \circ \iota$ of functions in $\mathbb{C}[\widehat{V}]$ with the matrix $[\iota]$. Since ι is a linear map, $D\iota$ is the matrix representation $[\iota]$. Since ι is injective, $[\iota]$ has maximal rank. For $DW|_{V^g}(c_1, \ldots, c_m) = (DW \circ \iota)(c_1, \ldots, c_m)[\iota]$ to be 0, we must have $DW \circ \iota(\vec{c}) = 0$. Since W is nondegenerate, we must have $\iota(\vec{c}) = 0$. Since ι is injective, $\vec{c} = 0$. Thus $W|_{V^g}$ is nondegenerate.

Now we show that $W|_{V^g}$ is quasihomogeneous. Let $V := \mathbb{C}^n$ and f_λ denote the $\mathbb{C}^*$ operator on V corresponding to W; recall that for $\lambda \in \mathbb{C}^*$, $W \circ f_\lambda = \lambda W$ and in the standard coordinates, $f_\lambda e_i = \lambda^{q_i} e_i$. For $W|_{V^g}$ to be quasihomogeneous, there must be a basis for V^g with corresponding weights $r_1, \ldots, r_k$ and corresponding $\mathbb{C}^*$ operator g_λ such that $W|_{V^g} \circ g_\lambda = \lambda W|_{V^g}$. To demonstrate such an operator we utilize the fact that commuting operators on a vector space are simultaneously diagonalizable. Since f_λ and g necessarily commute, by definition of G_W^{sm}, there is a basis of $\mathbb{C}^n$ on which both f_λ and g act diagonally. In particular, V^g decomposes into a direct sum of subspaces corresponding to weights of W: if $q_1, \ldots, q_n$ are the weights of W and $V^{q_1}, \ldots, V^{q_n}$ are the corresponding subspaces for each weight, $V^g = \sum_{q_i} V^g \cap V^{q_i}$. Hence f_λ restricted to V^g is still diagonal and for $v \in V^g$, $W(f_\lambda v) = \lambda^d W(v)$. That is, $W|_{V^g}$ is quasihomogeneous.

□

Since $W|_{V^g}$ is quasihomogeneous, nondegenerate, the Milnor ring $\mathbb{C}[s_1, \ldots, s_k]/JW|_{V^g}$ is well-understood. Moreover, we can keep the same definition for the unprojected g-sector as in the diagonal case.

Definition 4.3. For any symmetry of W (not just the diagonal ones)

$$\mathscr{H}_g := \mathscr{Q}_{W|_{V^g}} = \mathbb{C}[s_1, \ldots, s_k]/JW|_{V^g}.$$

Example 4.4. Consider P_8 and the group element (12). This has fixed locus spanned by $(1,1,0)$ and $(0,0,1)$. Let u and v be dual to $(1,1,0)$ and $(0,0,1)$, respectively. We calculate that $W|_{V^{(12)}} = 2u^3 + v^3$. Then $\mathscr{H}_g = \mathbb{C}[u,v]/\langle 6u^2, 3v^2 \rangle$. This has as a basis the elements $\lfloor 1, (12) \rceil$, $\lfloor u, (12) \rceil$, $\lfloor v, (12) \rceil$ and $\lfloor uv, (12) \rceil$.

4.2 Alternate $\mathscr{H}_g$

Now we develop some equivalent expressions of $\mathscr{H}_g$. This are easily seen to be equivalent for diagonal symmetries, but take more work to establish when we generalize. Rather then take the Milnor ring of $W|_{V^g}$, we take successive quotients of $\mathscr{H}_1 = \mathscr{Q}_W$. Explicitly, we quotient out the non-1 "eigenpolynomials" or eigenvectors of $\widehat{g}$. Let $m_1, \ldots m_n$ be n linearly independent eigenpolynomials of $\widehat{g}$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. Define

$$I_g := \langle m_i - \widehat{g}(m_i) \rangle,$$

$$\mathscr{H}_g' := \mathscr{Q}_W / I_g.$$

Note that in $\mathscr{H}_g'$ we really mean to quotient by the image of I_g in $\mathscr{Q}_W$. Also, $I_g = \langle (1-\lambda_i) m_i \rangle$.

Remark. We make the same remark as in the introduction, preceding proposition 4.15, on notation: if $m \in \mathbb{C}[\mathbf{x}]$, we use $\lfloor m, I \rceil$ to denote the residue of m in the quotient space $\mathbb{C}[x_1, \ldots, x_n]/I$. To be brief, $\lfloor m, g \rceil := \lfloor m, JW + I_g \rceil$. We will also use this notation to denote the image of an ideal in a quotient. Where context is clear, we drop the mention of the ideal.

Similar in look but different in meaning, we use square brackets to denote matrices. For example, $[T]_E^B$ will mean the matrix representation of the linear transformation $T : V \to W$

where E and B are bases for V and W, respectively. The bases are usually clear and so will be suppressed.

The following theorem motivates the notation $\lfloor m, g \rceil = \lfloor m, JW + I_g \rceil$:

Theorem 4.5. If R is a ring and I, J are ideals, $(R/I)/\lfloor J, I \rceil \cong R/(I+J)$.

Proof. Consider diagram 4.1.

$$\begin{tikzcd}
R \arrow[r, "\kappa_{IJ}"] \arrow[d, "\kappa_I"'] & R/(I+J) \\
R/I \arrow[ur, dashed, two heads, "\varphi"] \arrow[r, two heads, "\kappa_J"'] & (R/I)/\lfloor J, I \rceil \arrow[u, dashed, two heads, "\theta"']
\end{tikzcd}$$

Figure 4.1: A ring quotient diagram

Here κ_I, κ_{IJ}, and κ_J are quotient maps. Since $I \subset I+J$, the UMP of quotient rings induces a map $\varphi : \Omega^n/I \to \Omega^n/(I+J)$ such that $\varphi\kappa_I = \kappa_{IJ}$. Furthermore, φ is surjective since κ_{IJ} is surjective.

Now we use the UMP of $(R/I)/\lfloor J, I \rceil$. Let $\lfloor j, I \rceil \subset \lfloor J, I \rceil$. Since κ_I is surjective, $\lfloor j, I \rceil = \kappa_I(j)$ and $\varphi(\lfloor j, I \rceil) = \varphi(\kappa_I(j)) = \kappa_{IJ}(j) = 0$. Since $\lfloor J, I \rceil \subset \ker\varphi$, there is an induced (surjective, since φ is) map $\theta : (R/I)/\lfloor J, I \rceil \to \mathbb{R}/(I+J)$ so that $\theta\kappa_J = \varphi$. Let $\lfloor k, I \rceil \in \ker\varphi$. Then $\varphi(\kappa_I(k)) \subset I+J$, so $k = i+j \in I+J$. We have $\lfloor i+j, I \rceil = \lfloor j, I \rceil \subset \lfloor J, I \rceil$, or $\ker\varphi \subset \lfloor J, I \rceil$. This indicates that θ is injective.

Thus $(R/I)/\lfloor J, I \rceil \cong R/(I+J)$.

□

As a straightforward corollary, we get the following:

Corollary 4.6. $\mathscr{H}'_g \cong \mathbb{C}[\mathbf{x}]/(JW + I_g)$

Example 4.7. Let's do P_8 and (12) again, and calculate $\mathscr{H}'_g$. Here $JW = \langle 3x^2, 3y^2, 3z^2 \rangle$ and $I_g = \langle x - y \rangle$. We calculate $\mathscr{H}'_g = (\mathbb{C}[x,y,z]/JW)/I_g = \{1, x, y, z, xy, xz, yz, xyz\}/\langle x - y \rangle = \{1, x, z, xz\}$.

4.2.1 Establishing that the two expressions of $\mathscr{H}_g$ are the same. Before we prove that $\mathscr{H}_g$ is isomorphic to $\mathscr{H}'_g$, we establish an important pair of preliminary results about the coordinate independence of the unprojected g-sectors. For the next two lemmas, we need to keep track of a choice of coordinates. If $\{v_1, \ldots, v_k\}$ and $\{w_1, \ldots, w_k\}$ are two bases for V^g and $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_k\}$ are the corresponding dual bases, V^g_v or V^g_x will denote V^g and the choice of basis. The spaces V^g_v and V^g_w are the same, as are $\widehat{V^g_x}$ and $\widehat{V^g_y}$. However, $\mathbb{C}[V^g_x] = \mathbb{C}[x_1, \ldots, x_k]$ and $\mathbb{C}[V^g_y] = \mathbb{C}[y_1, \ldots, y_k]$ are different. We use this notation in expressions such as $W|_{g,x}$ to denote W expressed in terms of the x coordinates and restricted to V^g. There is a similar meaning to $JW|_{g,x}$. If $\{e_1, \ldots, e_n\}$ is a basis for V with dual basis $\{u_1, \ldots, u_n\}$, there is a similar meaning to W_u, JW_u, and $I_{g,u}$.

Lemma 4.8. *The Milnor ring $\mathscr{H}_g = \mathscr{Q}_{W|_{V^g}}$ is independent of the choice of basis for V^g; that is, if $\{v_1, \ldots, v_k\}$ and $\{w_1, \ldots, w_k\}$ are two bases for V^g, and $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_k\}$ are the respective corresponding dual bases, then $\mathbb{C}[x_1, \ldots, x_k]/JW|_{g,x}$ and $\mathbb{C}[y_1, \ldots, y_k]/JW|_{g,y}$ are isomorphic.*

Proof. Let $\{v_1, \ldots, v_k\}$ and $\{w_1, \ldots, w_k\}$ be two bases for V^g, and $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_k\}$ be the respective corresponding dual bases. There is an isomorphism $T : V^g \to V^g$ that is the change of basis transformation from v's to w's. That is, T is the identity on V and $v_i = t_{ji} w_j$. This gives an isomorphism $\widehat{T} : \widehat{V^g_y} \to \widehat{V^g_x}$, which extends to an isomorphism $\mathbb{C}[\mathbf{y}] \to \mathbb{C}[\mathbf{x}]$. To obtain an isomorphism on the quotients, we need a filler for the diagram in figure 4.2. In this diagram, $\kappa_{g,y}$ and $\kappa_{g,x}$ are the obvious quotient maps.

$$\begin{tikzcd}
\mathbb{C}[\mathbf{y}] \arrow[r, "\widehat{T}", two heads] \arrow[d, "\kappa_{g,y}"'] & \mathbb{C}[\mathbf{x}] \arrow[d, "\kappa_{g,x}"] \\
\mathbb{C}[\mathbf{y}]/JW_{g,y} \arrow[r, dashed] & \mathbb{C}[\mathbf{x}]/JW_{g,x}
\end{tikzcd}$$

Figure 4.2: The diagram for the coordinate independence of $\mathscr{H}_g$.

From the universal mapping property (UMP) of the quotient ring, the filler for diagram 4.2 exists if $JW_{g,y}$ is in the kernel of the map $\kappa_{g,x}\widehat{T}$. The filler is surjective since every other map in the diagram is surjective. Finally, the filler map is injective if $JW_{g,y}$ actually is the kernel of $\kappa_{g,x}\widehat{T}$. Thus, we establish the result if we show that $\widehat{T}^{-1}(JW|_{g,x})$ to be $JW_{g,y}$.

The ideal $JW_{g,x}$ is generated by the entries of the derivative matrix $D_xW|_{g,x}$. By the chain rule, we can express this in terms of the derivative matrix $D_yW_{g,y}$:f $W_{g,x} = W_{g,y} \circ T$, so $D_xW_{G,x} = D(W_{g,y} \circ T) = DW_{g,y} \circ T\ DT$. Since T is a linear transformation, DT is the matrix representation of T in terms of the v and w bases. This shows $DW_{g,x} = \widehat{T}(DW_{g,y}\ [T]$, or the derivatives of $W_{g,x}$ are linear combinations of the images of derivatives of $W_{g,y}$. Thus $JW_{g,x} \subset \widehat{T}(JW_{g,y})$. Since T is invertible, so is $[T]$ and it follows that $\widehat{T}^{-1}(JW_{g,x}) = JW_{g,y}$ $\square$

Lemma 4.9. *The quotient $\mathscr{H}_g' = \mathscr{Q}_W/I_g$ is independent of the choice of basis for V^g; that is, if $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_n\}$ are two bases for V and $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ are the corresponding dual bases, then $\mathbb{C}[x_1, \dots, x_n]/(JW + I_{g,x})$ and $\mathbb{C}[y_1, \dots, y_n]/(JW + I_{g,y})$ are isomorphic.*

Proof. The proof is similar to Lemma 4.8. Let $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_n\}$ be two bases for V, and $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ be the respective corresponding dual bases. Let T be the identity map on V. The matrix $[T]_v^w$ has entries t_{ij} so that $v_i = t_{ji}w_j$. This gives a map on $\widehat{V}$ that can be written in terms of the entries of $[T]$ as $y_{ij} = t_{ij}x_j$. This is significant because it gives an isomorphism $\widehat{T} : \mathbb{C}[\mathbf{y}] \to \mathbb{C}[\mathbf{x}]$.

$$\begin{tikzcd}
\mathbb{C}[\mathbf{y}] \arrow[r, two heads, "\widehat{T}"] \arrow[d, two heads, "\kappa_{g,y}"'] & \mathbb{C}[\mathbf{x}] \arrow[d, two heads, "\kappa_{g,x}"] \\
\mathbb{C}[\mathbf{y}]/(JW_y + I_{g,y}) \arrow[r, dashed, two heads] & \mathbb{C}[\mathbf{x}]/(JW_x + I_{g,x})
\end{tikzcd}$$

Figure 4.3: The diagram for the coordinate independence of $\mathscr{H}_g'$.

To obtain an isomorphism on the quotients, we need a filler for the diagram in figure 4.3. In this diagram, $\kappa_{g,y}$ and $\kappa_{g,x}$ are the obvious quotient maps. As in Lemma 4.8, the filler exists and is an isomorphism on the quotients if $T^{-1}(JW_s + I_{g,s}) = JW_y + I_{g,y}$.

Now we show $\widehat{T}(I_g) = I_{g,y}$. First, recall that $I_g = \langle(1-\lambda_i)m_i\rangle$ where the λ_i are eigenvalues of $\widehat{g}$ and the m_i are the corresponding eigenvectors of $\widehat{g}$ in $\mathbb{C}[\mathbf{x}]$. From Proposition 4.1, the m_i have the form $m_{ij}^{-1}x_j$ where the eigenvector of g corresponding to λ_i is $m_{ji}v_j$.

We can relate $\widehat{g}_y$ to $\widehat{g}$ via coordinate transformations. Let $T : V \to V$ be the identity transformation on V with matrix $[T]_v^w$. That is, $v_i = t_{ji}w_i$. Then $\widehat{T}$ is a ring map from $\mathbb{C}[V_w] = \mathbb{C}[y_1, \ldots, y_n]$ to $\mathbb{C}[V_v] = \mathbb{C}[x_1, \ldots, x_n]$. Using the matrix of T, we find that the map $\widehat{g}_y$ on $\mathbb{C}[\mathbf{y}]$ is given by $y_i \mapsto t_{ij}g_{jk}t_{kl}^{-1}y_l$.

We'll show that $\widehat{T}^{-1}(m_i)$ is an eigenvector of $\widehat{g}_y$ with the same eigenvalue as m_i by direct computation:

$$\begin{aligned}
\widehat{g}_y(\widehat{T}^{-1}(m_i)) &= \widehat{g}_y(\widehat{T}^{-1}(m_{ij}^{-1}x_j)) \\
&= \widehat{g}_y(m_{ij}^{-1}t_{jk}^{-1}y_k) \\
&= m_{ij}^{-1}t_{jk}^{-1}t_{kl}g_{lm}t_{mn}^{-1}y_n \\
&= m_{ij}^{-1}\delta_{jl}g_{lm}t_{mn}^{-1}y_n \\
&= m_{il}^{-1}g_{lm}t_{mn}^{-1}y_n.
\end{aligned}$$

We need the identity $m_{ij}^{-1}g_{jk} = \lambda_i^{-1}m_{ik}$, which follows since $g(m_{ji}v_j) = m_{ji}g_{kj}v_k = \lambda_i m_{ik}v_k$ eigenvalue-eigenvector relation of g. Using this identity to continue, we have:

$$\begin{aligned}
\widehat{g}_y(\widehat{T}^{-1}(m_i)) &= \lambda_i m_{im}^{-1}t_{mn}^{-1}y_n \\
&= \lambda_i\widehat{T}^{-1}(m_{im}^{-1}x_m) \\
&= \lambda_i\widehat{T}^{-1}(m_i).
\end{aligned}$$

Thus $\widehat{T}$ takes eigenvectors of $\widehat{g}_x$ to eigenvectors of $\widehat{g}_y$ with the same eigenvalues. Since $\widehat{T}$ is an isomorphism, $I_{g,x} = \widehat{T}(I_{g,y})$.

Similarly to Lemma 4.8, we have that $JW_y = \widehat{T}^{-1}(JW)$. This follows from examining the derivatives, or rather the derivative matrices, in each coordinate system. The chain rule gives us $DW_y = D(W \circ T) = \widehat{T}(DW)[T]$. Since T is invertible, $[T]$ is also invertible and we have $JW_y = \widehat{T}^{-1}(JW)$.

Since $I_{g,y} = \widehat{T}^{-1}(I_{g,x})$ and $JW_y = \widehat{T}^{-1}(JW)$, we have $T^{-1}(JW_s + I_{g,s}) = JW_y + I_{g,y}$ and so obtain the desired isomorphism.

$\square$

Remark. The coordinate independence of $\mathscr{H}_g$ and $\mathscr{H}_g'$ justifies the notation $\mathbb{C}[\widehat{V}]/(JW + I_g)$ and $\mathbb{C}[\widehat{V^g}]/JW_g$, which doesn't indicate a choice of basis or indeterminates.

Theorem 4.10. *$\mathscr{H}_g \cong \mathscr{H}_g'$ as rings.*

Proof. Let $g \in G$ be a possibly nondiagonal symmetry. By Lemmas 4.8 and 4.9, it is sufficient to show $\mathscr{H}_g \cong \mathscr{H}_g'$ in one coordinate system. It is most convenient to choose coordinates $e_1, \ldots, e_n$ on which g acts diagonally. Again, we know g is diagonal since g is a symmetry with finite order (see [Koo03]). Furthermore, order the basis for V so that the first k vectors $e_1, \ldots e_k$ are in the 1-eigenspace. That is, $\{e_1, \ldots, e_k\}$ is a basis for V^g. Let $\{s_1, \ldots, s_n\}$ be the corresponding dual basis to $\{e_1, \ldots, e_n\}$ (so $\{s_1, \ldots, s_k\}$ is a basis for $\widehat{V^g}$).

Let $\iota : V^g \hookrightarrow V$ be the inclusion map and $\widehat{\iota} : \mathbb{C}[\widehat{V}] \to \mathbb{C}[\widehat{V^g}]$ be the ring map induces by the adjoint of ι. As in the lemmas on coordinate independence, we need a filler to the diagram in figure 4.10. The filler exists and is an isomorphism if $\widehat{\iota}^{-1}(JW_g) = JW + I_g$.

$$\begin{tikzcd}
\mathbb{C}[\widehat{V}] \arrow[r, two heads, "\widehat{\iota}"] \arrow[d, "\kappa_g"'] & \mathbb{C}[\widehat{V^g}] \arrow[d, "\kappa_g"] \\
\mathbb{C}[\widehat{V}]/(JW + I_g) \arrow[r, dashed, two heads] & \mathbb{C}[\widehat{V^g}]/JW_g
\end{tikzcd}$$

Figure 4.4: The diagram for the isomorphism between $\mathscr{H}_g$ and $\mathscr{H}_g'$.

Observe that $\widehat{\iota}(s_i) = \begin{cases} s_i & i \leq k \\ 0 & i > k \end{cases}$. Since $I_g = \langle (1 - \lambda_i)s_i \rangle$ is generated by the variables s_j for $j \geq k$, clearly $\widehat{\iota}(I_g) = \langle 0 \rangle$. Also, if $m \in \mathbb{C}[\mathbf{s}]$ is an element of I_g, then m has a factor s_j for some $j > k$. Then $\widehat{\iota}(m) = 0 \in JW_g$. Thus $JW + I_g \subset \widehat{\iota}(JW_g)$.

Say $\widehat{\iota}(m) \in JW_g$ for some monomial $m \in \mathbb{C}[\mathbf{s}]$. If $\widehat{\iota}(m)$ is 0, then m has a factor of s_j for some $j > k$ and $m \in I_g$. If $\widehat{\iota}(m)$ is not 0, then $\widehat{\iota}(m) = m$. This means $m \in JW_g = \langle \frac{\partial W}{\partial s_1}, \ldots, \frac{\partial W}{\partial s_k} \rangle \subset \langle \frac{\partial W}{\partial s_1}, \ldots, \frac{\partial W}{\partial s_n} \rangle = JW$. Thus $\widehat{\iota}^{-1}(JW_g)$. Combined with the previous observation, we have $JW + I_g \subset \widehat{\iota}(JW_g)$.

Hence $\widehat{\iota}^{-1}(JW_g) = JW + I_g$, and the map $\mathscr{H}'_g \to \mathscr{H}_g$ is an isomorphism. □

Example 4.11. Note the clear isomorphism between examples 4.4 and 4.7.

4.2.2 More ways to understand $\mathscr{H}'_g$. We end the section stating a few more facts about I_g and their implications concerning $\mathscr{H}'_g$. For example, $\mathscr{H}'_g$ can also be understood as

$$\frac{\mathbb{C}[\mathbf{x}]/JW}{\langle x_i - \widehat{g}(x_i) \rangle} \text{ or } \frac{\mathbb{C}[\mathbf{x}]}{(JW + \langle x_i - \widehat{g}(x_i) \rangle)}.$$

This comes from the following fact:

Fact 4.12. $I_g = \langle x_i - \widehat{g}(x_i) \rangle$

Proof. Since $\widehat{g}$ is diagonalizable and the n eigenpolynomials $m_1, \ldots, m_n$ span P_1, $P_1 = M_1 \oplus M'$ where M_1 is the 1-eigenspace of $\widehat{g}$ and M' is the sum of the non-1 eigenspaces. Since $x_i \in P_1$, $x_i = \sum_j m_j \in M_1 \oplus M'$. $x_i - \widehat{g}(x_i) = \sum m_j - \left(\widehat{g}(m_1) + \sum_{j \neq 1} \widehat{g}(m_j) \right) = (m_1 - m_1) + \sum (1 - \lambda_j) m_j$, so $x_i - \widehat{g}(x_i) \in I_g$ for all i.

Conversely, let m_i be an eigenpolynomial of $\widehat{g}$ with eigenvalue λ_i. Write $m_i = \sum_j^n a_j x_j$ where $a_j \in \mathbb{C}$. Then $m_i - \widehat{g}(m_i) = \sum a_j x_j - \sum a_j (x_j \circ g) = \sum a_j (x_j - (x_j \circ g))$, so $m_i \in \langle x_i - \widehat{g}(x_i) \rangle$ for all i. □

Another expression of $\mathscr{H}'_g$ that we use later is:

$$\mathscr{H}'_g = \mathscr{Q}_W / \langle y - \widehat{g}(y) \rangle.$$

This follows from the useful fact:

Fact 4.13. $I_g = I_\Omega := \langle y - \widehat{g}(y) : y \in \mathbb{C}[\mathbf{x}]\rangle$.

Proof. Clearly $I_g \subset I_\Omega$.

Now we show that $\langle y - \widehat{g}(y) : y \in \mathbb{C}[\mathbf{x}]\rangle \subset I_g$.

First, we show that $(x_i^n - (x_i^n) \circ g) \in I_g$. Note that

$$\begin{aligned}
(a^{n-1} + a^{n-2}\widehat{g}(a) + \ldots + a\widehat{g}^{n-2}(a) + \widehat{g}^{n-1}(a))(a - \widehat{g}(a)) \\
= (a^n - a^{n-1}\widehat{g}(a) + a^{n-1}\widehat{g}(a) - a^{n-2}\widehat{g}(a)\widehat{g}(a) + \\
\ldots + a^2\widehat{g}^{n-2}(a) - a\widehat{g}^{n-2}(a)\widehat{g}(a) + a\widehat{g}^{n-1}(a) \\
-\widehat{g}^{n-1}(a)\widehat{g}(a)) \\
= (a^n - \widehat{g}^n(a)) \\
= (a^n - \widehat{g}(a^n))
\end{aligned}$$

for $a \in \mathbb{C}[\mathbf{x}]$. We used liberally the fact that $\widehat{g}$ is a ring homomorphism, so $\widehat{g}(a)\widehat{g}(b) = \widehat{g}(ab)$. In particular, this shows $x_i^k - \widehat{g}(x_i^k) \in I_g \forall i$.

Next we show that $(x_i x_j - \widehat{g}(x_i x_j)) \in I_g$. For$a, b \in \mathbb{C}[\mathbf{x}]$, we compute

$$\begin{aligned}
(a + \widehat{g}(a))(b - \widehat{g}(b)) + (b + \widehat{g}(b))(a - \widehat{g}(a)) \\
= (ab - a\widehat{g}(b) + b\widehat{g}(a) - \widehat{g}(ab) + ab + a\widehat{g}(b) - b\widehat{g}(a) + \widehat{g}(ab)) \\
= (2ab - 2\widehat{g}(ab))
\end{aligned}$$

If $a = x_i, b = x_j$, we get that $x_i x_j - \widehat{g}(x_i x_j) \in I_g$.

Combining these two results, we get that for all y in $\mathbb{C}[\mathbf{x}]$, the polynomial $y - \widehat{g}(y)$ is an element of I_g. This gives us that $I_\Omega \subset I_g$. Combined we the reverse inclusion as shown earlier, $I_g = I_\Omega$.

□

In our effort to generalize the construction of $\bigoplus_{g \in G} \mathscr{H}_G$, which contains the G-invariant subspace $\mathscr{B}_W^G$, we now have the g sector $\mathscr{H}_g$ on a firm footing. Most often, we understand this as defined, $\mathbb{C}[\widehat{V^g}]/JW|_{V^g}$. The other expressions are convenient because they yield convenient quotient maps, as we see later in this thesis.

4.3 $\bigoplus_{g\in G} \mathscr{H}_g$ IS A $\mathbb{C}G$-MODULE

The next step in our progress toward a graded Frobenius algebra with a group action is to define the group action. There is an obvious way for automorphisms to act on $\mathbb{C}[x_1,\ldots,x_n]$, but this is not the action we need for our constructions. We need a "determinant-twisted" G-action which we define first on $\mathscr{H}_1 = \mathscr{Q}_W = [\mathbf{x}]/JW$ and then on an arbitrary g-sector $\mathscr{H}_g$. We define the group action on $\bigoplus_{g\in G} \mathscr{H}_g'$ and then on $\bigoplus_{g\in G} \mathscr{H}_g$, and end this section by showing that these spaces are isomorphic as $\mathbb{C}G$-modules.

Definition 4.14. Let G be a subset of G_W^{sm}, g be an element of G, and m be an element of $\mathscr{H}_1 = \mathscr{Q}_W = \mathbb{C}[\mathbf{x}]/JW$. The action of h on $\lfloor m, 1\rceil$ is given by

$$\lfloor m, JW\rceil \cdot h \quad = \quad \lfloor m \circ h\,\det(g), JW\rceil.$$

Proposition 4.15. *The above operation is a right G-action on $\mathbb{C}[x_1,\ldots,x_n]/JW$.*

Proof. The action of g on $\mathbb{C}[x_1,\ldots,x_n]/JW$ is $(\lfloor f, JW\rceil)\cdot g := \lfloor f\cdot g, JW\rceil$. This action is well-defined if $JW\cdot g \subset JW$. We show that, in fact, $JW = JW\cdot g$. It will be key to the proof that g has finite order, but this is no concern as we assume our automorphisms have finite order throughout most of this thesis.

We apply the chain rule to the functions $W : \mathbb{C}^n \to \mathbb{C}$ and $g : \mathbb{C}^n \to \mathbb{C}^n$. We get $D(W\circ g)(x) = DW(g(x))\cdot Dg(x)$. Here the $\cdot$ means matrix multiplication, since DW is a $1\times n$ matrix and Dg is an $n\times n$ matrix. $Dg(x) = [g]$ since g is a linear transformation. Observe that $(\dfrac{\partial W}{\partial x_i})\widehat{g} = \dfrac{\partial W}{\partial x_i}\circ g$, so g acting on DW is $DW\circ g = DW(g(x))$. We have $D(W\circ g) = \widehat{g}(DW)\cdot[g]$. Since $W = W\circ g$, $DW = D(W\circ g) = \widehat{g}(DW)\cdot[g]$. This tells us that any partial of W is a linear combination of partials of $\widehat{g}(DW)$, so $JW \subset \widehat{g}(JW)$.

We apply this procedure further: $W \circ g = W \circ g^{k-1}$, so $D(W \circ g) = D(W \circ g^k) = D(W \circ g)(g^{k-1}(x)) \cdot Dg^{k-1}(x)$. From above, $D(W \circ g) = \widehat{g}(DW) \cdot [g]$. We have $\widehat{g}^{k-1}(D(w \circ g)) \cdot [g]^{k-1} = \widehat{g}^{k-1}(\widehat{g}(DW) \cdot [g]) \cdot [g]^{k-1} = (\widehat{g}^k(DW)) \cdot [g]^k$, since g is linear. This gives $\widehat{g}(DW) \cdot [g] = \widehat{g}^k(DW) \cdot [g]^k$. We can undo the $[g]$-combination, since $[g]$ is invertible with entries in $\mathbb{C}$, to obtain $\widehat{g}(DW) = \widehat{g}^k(DW) \cdot [g]^{k-1}$. Thus $\widehat{g}(JW) \subset \widehat{g}^k(JW)$. Since g has finite order, $\widehat{g}(JW) \subset \widehat{g}^{|g|}(JW) = JW$.

For the proposed operation to be a group action, (we apologize for the overloading of $\cdot$, which means group action here) we need $\lfloor m, JW \rceil \cdot 1 = \lfloor m, JW \rceil$ and $(\lfloor m, JW \rceil \cdot h) \cdot k = \lfloor m, JW \rceil \cdot (hk)$. The first condition is clear. The second is straightforward:

$$\begin{aligned}
(\lfloor m, JW \rceil \cdot h) \cdot k &= \lfloor (m \circ h \det(h)), JW \rceil \cdot k \\
&= \lfloor (m \circ h \circ k \det(h)\det(k)), JW \rceil \\
&= \lfloor (m \circ (hk)\det(hk)), JW \rceil \\
&= \lfloor m, JW \rceil \cdot hk.
\end{aligned}$$

□

The proof illustrates why the action is necessarily a right G-action.

Example 4.16. Consider again P_8 and the symmetry group elements $(12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $(123) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. Let's calculate what (12) and (123) do to some elements of $\mathcal{Q}_W$.

$$\lfloor 1, JW \rceil \cdot (12) = \lfloor \det(12), JW \rceil = -\lfloor 1, JW \rceil$$

$$\lfloor x, JW \rceil \cdot (12) = \det(12) \lfloor \widehat{(12)}(x), JW \rceil = -\lfloor y, JW \rceil$$

Finally,

$$\lfloor x+2y+3z, JW\rceil \cdot (123) = \det(123)\lfloor\widehat{(123)}(x+2y+3z), JW\rceil = \lfloor z+2x+3y, JW\rceil.$$

4.3.1 Extending the group action to $\mathscr{H}_g$. Here we extend the action on $\mathscr{H}_1$ to $\mathscr{H}_g$. This will make $\bigoplus_{g\in G}\mathscr{H}_g$ a right $\mathbb{C}G$- module, meaning it is a $\mathbb{C}$-vector space with a right G-action. (Note that each $\mathscr{H}_g$ is already a vector space, so $\bigoplus \mathscr{H}_g$ is as well.) Then we define a similar action on $\mathscr{H}'_g$, and show that the isomorphism of Theorem 4.10 between $\mathscr{H}_g$ and $\mathscr{H}'_g$ respects the group actions.

In the diagonal case, a group element h acts on $m \in \mathscr{H}_g$ by $m \cdot h := \det(h|_{V^g})m \circ h$. The issue in generalizing this is that not every $h \in G$ is an operator on V^g, so does not have a determinant when restricted to V^g.

For the action of h on $\mathscr{H}_g$ to "twist" by the determinant of $h \in G$ a symmetry group, we need an appropriate space corresponding to g whereon h is a linear operator with a well-defined determinant. Since arbitrary group elements move sectors to conjugate sectors, the right space is the direct sum over elements of the conjugacy class of g of the fixed loci of. Let $\mathcal{K}_g$ be the conjugacy class of g. Define $V^{(g)}$ to be the space $\bigoplus_{k\in\mathcal{K}_g} V^k$. For h in G, h takes things in each summand to another conjugate of g, so h restricts to a linear operator on $V^{(g)}$. Then we have a meaningful restriction of h to a linear operator involving the fixed locus of g. This lets us define the appropriate "determinant twisted" group action.

Remark. Let W be a quasihomogeneous, nondegenerate polynomial and $g, h \in G_W^{sm}$. While h is a map from V to V, $h|_{V^g}$ is a different map with a different determinant and range. We find shortly that the range of $h|_{V^g}$ is $V^{h^{-1}gh}$. To denote the restriction of h to V^g, we will use the notation h_g or, if it is useful to keep track of the range, $h_g^{h^{-1}gh}$.

Also, if $\mathcal{K}_g$ is the conjugacy class of g, it is useful to define $V^{(g)} := \bigoplus_{k\in\mathcal{K}_g} V^k$ and $h_{(g)}$ to be h restricted to $V^{(g)}$.

Now we present the G-action on $\bigoplus_{g\in G}\mathscr{H}'_g$. For $g, h \in G$ a subgroup of G_W^{sm}, let $h_{(g)}$ be

h restricted to $V^{(g)}$. and $\lfloor f,g\rceil \in \mathscr{H}'_g$, define

$$(\lfloor f,g\rceil)\cdot h \quad := \quad \lfloor \det(h_{(g)})f\circ h, h^{-1}gh\rceil.$$

Proposition 4.17. *This action is a well-defined right group action on $\bigoplus_{g\in G}\mathscr{H}'_g$.*

Proof. To verify this action is well-defined, suppose $\lfloor a,g\rceil = \lfloor b,g\rceil$, or $a(x) = b(x)+j(x)+m(x)$ where $j\in JW$ and $m\in I_g$.

We have $\lfloor a,g\rceil\cdot h = \lfloor \det(h)_{(g)}b\circ h, h^{-1}gh\rceil + \lfloor \det(h_{(g)})j\circ h, h^{-1}gh\rceil + \lfloor \det(h_{(g)})m\circ h, h^{-1}gh\rceil$. We established earlier (see Proposition 4.15 in the beginning of the chapter) that $j\circ h\subset JW$ for all $h\in G$, so $\lfloor j\circ h, h^{-1}gh\rceil = 0$. Now the group action is well-defined if $\lfloor \det(h_{(g)})m\circ h, h^{-1}gh\rceil = 0$. By linearity of h, this follows if $((1-\lambda_i)m_i)\cdot h\in I_{g^{-1}hg}$ for all eigenpolynomials m_i of $\widehat{g}$.

It suffices to show that $\widehat{h^{-1}gh}(m_i)\in I_{h^{-1}gh}$ for m_i an eigenvector of $\widehat{g}$. Observe a simple fact about eigenvectors, adjoints, and conjugate operators: If $\widehat{g}(m)=\lambda m$, then $\widehat{h^{-1}gh}(m\circ h) = \widehat{h}\widehat{g}\widehat{h^{-1}}\widehat{h}(m) = \lambda\widehat{h}(m)$. Thus $m_i\circ h$ is an eigenvector of $\widehat{h^{-1}gh}$ with eigenvalue λ_i, so $\det(g)(1-\lambda_i)\det(h^g_{h^{-1}gh})m_i\circ h\in I_{h^{-1}gh}$. The G-action is well-defined.

To check that we have a group action, we need to satisfy the conditions: $\lfloor f,g\rceil\cdot 1 = \lfloor f,g\rceil$ and $((\lfloor f,g\rceil)\cdot h)\cdot k = (\lfloor f,g\rceil)\cdot hk$. The first is clear; the second is straight-forward:

$$\begin{aligned}
((\lfloor f,g\rceil)\cdot h)\cdot k &= (\lfloor \det(h_{(g)})\ f\circ h, h^{-1}gh\rceil)\cdot k\\
&= \det(h_{(g)})\lfloor \det(k_{(h^{-1}gh)})\ (f\circ h)\circ k, k^{-1}h^{-1}g(hk)\rceil\\
&= \det(h_{(g)})\det(k_{(g)})\lfloor f\circ(hk), (hk)^{-1}g(hk)\rceil\\
&= \det(hk_{(g)})\lfloor f\circ(hk)\rceil, (hk)^{-1}g(hk)\\
&= \lfloor f,g\rceil\cdot(hk).
\end{aligned}$$

We used the fact that $h^{-1}gh$ is in the same conjugacy class as (g), so $k_{(h^{-1}gh)} = k_{(g)}$, as well as the fact that the determinant is multiplicative.

□

Here we see why our action is necessarily a right G-action; it is because the action requires pre-composition or composition on the right with the group element.

4.3.2 $\mathscr{H}_g \cong \mathscr{H}_{k^{-1}gk}$ **(and** $\mathscr{H}'_g \cong \mathscr{H}'_{k^{-1}gk}$**).** If we continue with the above proof, we establish a useful relation among g-sectors. It turns out that conjugate group elements have isomorphic sectors (isomorphic as vector spaces). If $h = k^{-1}gk$, then $\mathscr{H}_h \cdot k = \mathscr{H}_g$. We show that the linear transformation $T : \mathscr{H}_g \to \mathscr{H}_h$ given by $T(\lfloor m, g\rceil) := \lfloor m, g\rceil \cdot k$ is in fact a $\mathbb{C}$-vector space isomorphism. We have just shown in the preceding section that T is well-defined.

Now we show that T is bijective by considering the inverse map, acting by k^{-1}. That this is an inverse map follows readily from the fact that we have a well-defined group action, so

$$\begin{aligned} (\lfloor f, g\rceil \cdot k) \cdot k^{-1} &= \lfloor f, g\rceil \cdot (kk^{-1}) \\ &= \lfloor f, g\rceil \cdot 1 \\ &= \lfloor f, g\rceil. \end{aligned}$$

Similarly, $(\lfloor m, h\rceil \cdot k^{-1}) \cdot k = \lfloor m, h\rceil$.

4.3.3 G**-action on** $\bigoplus \mathscr{H}_g$**.** Similarly, we define the G-action on $\mathscr{H}_g = \mathbb{C}[\widehat{V^g}]/JW|_{V^g}$ by

$$\lfloor m, g\rceil \cdot h = \lfloor \det(h_{(g)}) m \circ h^g_{V^{h^{-1}gh}}, h^{-1}gh\rceil.$$

Proposition 4.18. *The proposed action on* $\mathscr{H}_g$ *is well-defined.*

The verification is similar to the case of $\bigoplus \mathscr{H}'_g$, so we refer the reader to Proposition 4.17.

Theorem 4.19. *The sum of isomorphisms of Theorem 4.10 is a $\mathbb{C}G$-module isomorphism between $\bigoplus \mathscr{H}_g$ and $\bigoplus \mathscr{H}'_g$.*

Proof. If we call the isomorphism $\mathscr{H}'_g \to \mathscr{H}_g$ by the name φ, it suffices to show that $\varphi(\lfloor m, g\rceil \cdot h) = \varphi(\lfloor m, g\rceil) \cdot h$.

On the one hand,

$$\begin{aligned}
\varphi(\lfloor m, g\rceil \cdot h) &= \varphi(\lfloor \det(h_{(g)}) m \circ h, h^{-1}gh\rceil) \\
&= \widehat{\iota_{h^{-1}gh}}(\lfloor \det(h_{(g)}) m \circ h, h^{-1}gh\rceil) \\
&= \lfloor \det(h_{(g)}) m \circ h \circ \iota_{h^{-1}gh}, h^{-1}gh\rceil
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\varphi(\lfloor m, g\rceil) \cdot h &= \widehat{\iota_g}(\lfloor m, g\rceil) \cdot h \\
&= \lfloor m \circ \iota_g, g\rceil \cdot h \\
&= \lfloor \det(h_{(g)}) m \circ \iota_g \circ h^g_{h^{-1}gh}, h^{-1}gh\rceil
\end{aligned}$$

We finish by noting that $m \circ h \circ \iota_{h^{-1}gh} = m \circ h_{h^{-1}gh} = m \circ \iota_g \circ h^g_{h^{-1}gh}$.

Thus $\varphi(\lfloor m, g\rceil \cdot h) = \varphi(\lfloor m, g\rceil) \cdot h$.

$\square$

4.4 What is the grading?

The state space $\mathscr{H}$ is graded in the nondiagonal case much as in the diagonal case. The definitions are mainly the same, but take more thought to justify. As before, the structure $\mathscr{H} := \bigoplus_{g \in G} \mathscr{H}_g$ is naturally graded by the group G. Additionally, the multiplication must respect bi-grading as defined in the diagonal case.

Recall that, for $\alpha \in \mathscr{H}_g$ with a well-defined weighted degree, we can give α a bi-degree as follows. That is, α has two distinct degrees which we label with plus and minus subscripts:

$$(\deg_+\lfloor\alpha, g\rceil, \deg_-\lfloor\alpha, g\rceil)_B := (\deg\alpha, \deg\alpha) + (\text{age } h, \text{age } h^{-1}) - (\sum_{i=1}^{n} q_i, \sum_{i=1}^{n} q_i).$$

We need to say what the weighted degree and age mean in the nondiagonal context, but the definition holds.

4.4.1 Weighted degree. Here we define the weighted degree. Let W be a quasihomogeneous, nondegenerate polynomial, $\{q_1, \ldots, q_n\}$ its weight system, and f_λ the corresponding $\mathbb{C}^*$ operator. Since f_λ is a diagonal operator, $f_\lambda(e_i) = \lambda^{q_i}e_i$. For any monomial $m = x_1^{a_1}\cdots x_n^{a_n}$, $m \circ f_\lambda = (\lambda^{q_1}x_1)^{a_1}\ldots(\lambda^{q_n}x_n)^{a_n} = \lambda^{a_1q_1+\ldots a_nq_n}x_1^{a_1}\cdots x_n^{a_n} = \lambda m$. We generalize this notion: say $p \in \mathbb{C}^[\mathbf{x}]$ and $p \circ f_\lambda = \lambda^q p$. Then we say that the *weighted degree* of p is q. Let $R_q = \{p \in \mathbb{C}[\mathbf{x}] :\ p \circ f_\lambda = \lambda^q p\}$. Since every element of $\mathbb{C}[\mathbf{x}]$ is a sum of monomials, which have a weighted degree, $\mathbb{C}[\mathbf{x}] = \bigoplus_{q\in\mathbb{Q}} R_q$.

It is a useful fact that the Milnor ring $\mathscr{Q}_W$ of a quasihomogeneous, nondegenerate polynomial W is finitely generated by elements with monomial representatives. This suggests that $\mathscr{Q}_W$ might finitely decompose by weighted degree; we now demonstrate that this is the case.

Proposition 4.20. *For W a quasihomogeneous, nondegenerate polynomial, $\mathscr{Q}_W$ finitely decomposes by weighted degree.*

Proof. The detail left to verify is ensuring that f_λ is well-defined on $\mathscr{Q}_W$. This is the case if $\widehat{f_\lambda}(JW) \subset JW$. Since JW is generated by the partial derivatives of W, examine what f_λ does to $\frac{\partial W}{\partial x_i}$. Further, we look by monomials $m = x_1^{a_1}\cdots x_n^{a_n}$ of W. Since $W \circ f_\lambda = \lambda W$, $m \circ f_\lambda = \lambda m$ for each monomial of W. If $a_i = 0$, m vanishes in $\frac{\partial W}{\partial x_i}$. If $a_i \geq 1$, then $\frac{\partial W}{\partial x_i}$ has a corresponding monomial $x_1^{a_1}\cdots x_i^{a_i-1}\cdots x_n^{a_n}$ with one power less of x_i. We get $m \circ f_\lambda =$

$(\lambda^{q_1}x_1)^{a_1}\dots(\lambda^{q_i}x_i)^{a_i-1}(\lambda^{q_n}x_n)^{a_n} = \lambda^{a_1q_1+\dots+(a_i-1)q_i\dots a_nq_n} = \lambda^{1-q_i}m$. Then $\frac{\partial W}{\partial x_i}\circ f_\lambda = \lambda^{1-q_i}\frac{\partial W}{\partial x_i}$. We have that $\widehat{f}_\lambda(\frac{\partial W}{\partial x_i}) \subset JW\ \forall i$. The partials generate JW, so $JW \circ f_\lambda \subset JW$.

Since f_λ is well-defined on $\mathscr{Q}_W$, meaning it is well-defined on elements with monomial representatives, and $\mathscr{Q}_W$ has a finite basis of monomials, we conclude that $\mathscr{Q}_W$ is graded by weighted degree into finitely many summands.

□

Since each symmetry group element g commutes with the $\mathbb{C}^*$ operator, g and f_λ simultaneously diagonalize. When we restrict to the fixed locus V^g of g, there is a basis of V^g for which $f_\lambda|_{V^g}$ acts diagonally. Furthermore, since $W|_{V^g}$ is quasihomogeneous, nondegenerate by 4.2 and $\mathscr{H}_g$ is the Milnor ring of $W|_{V^g}$, the above proposition indicates that $\mathscr{H}_g$ decomposes by weighted degree.

In the diagonal case, the mirror symmetry isomorphisms between $\mathscr{A}$- and $\mathscr{B}$-models required the weighted degree to reflect the dimension of V^g and the weights of $W|_{V^g}$ (see [Kra10]. To do this in the nondiagonal extension, we shift our grading by the weights. Let $W|_{V^g}$ have weight system $\{q_1,\dots,q_r\}$. On $\mathscr{H}_g$, let $\mathscr{H}_{q+\sum q_i} := \{\alpha \in \mathscr{H} :\ \widehat{f}_\lambda|_{V^g}(\alpha) = \lambda^q\alpha\}$. Thus $\mathscr{H} = \bigoplus_{q\in\mathbb{Q}}\mathscr{H}_n = \bigoplus_{g\in G}\mathscr{H}_g$. We can go further, and decompose each sector by degree. Let $\mathscr{H}_{g,q} := \{\alpha \in \mathscr{H}_g :\ \widehat{f}_\lambda|_{V^g}(\alpha) = \lambda^q\alpha\}$; then $\mathscr{H} = \bigoplus_{G,\mathbb{Q}}\mathscr{H}_{g,q}$. The weighted degree of an element α, if defined, is the number q such that $\alpha \in \mathscr{H}_q$. Note that $\deg\lfloor 1, g\rceil = \sum q_i$.

4.4.2 The rest of the bi-grading. The bi-grading requires the *age* of a group element; this comes from the eigenvalues of h. For each eigenvalue λ of h, let $\theta := \dfrac{1}{2\pi i}\log(\lambda)$, where we take the principal branch of the log here. Since each eigenvalue is a root of unity (see [Koo03]), each θ is a rational number in $[0, 2\pi)$. We call the θ_i the *phases* of h. The *age* of h is the sum of its phases; age $h = \sum_{i=1}^n \theta_i$.

Since the weighted degree of $\alpha \in \mathscr{H}_g$ and the age of $g \in G_W^{sm}$ are now well-defined, the bi-grading definition of the diagonal case is easily computable.

We make a few remarks about the bi-grading. If a single degree is given in the literature, it is the sum of the bi-degrees. If we sum, we get $N_h + 2\sum_{i=1}^n(\theta_i - q_i)$. Note that for elements in the identity sector, the $\mathscr{B}$ bi-degree reduces to the original weighted degree before we introduced the determinant- or weight-twisting. Finally, the $\mathscr{A}$-model bi-grading as presented in the nondiagonal case is the same as the diagonal case.

Example 4.21. Let $W = x^2y + y^3$. We computed $G_W^{sm} \cap SL$ already, it is isomorphic to $\mathbb{Z}_3$. Let a and a^2 denote the non-identity elements. These have trivial fixed loci, so $\bigoplus \mathscr{H}_g$ has as a basis

$$\{\lfloor 1, 1\rceil, \lfloor x, 1\rceil, \lfloor y, 1\rceil, \lfloor y^2, 1\rceil, \lfloor 1, a\rceil, \lfloor 1, a^2\rceil\}.$$

The bi-degrees of these basis elements are:

basis element	$\lfloor 1, 1\rceil$	$\lfloor x, 1\rceil$	$\lfloor y, 1\rceil$
bi-degree	(0,0)	$(1/3, 1/3)$	$(1/3, 1/3)$
basis element	$\lfloor y^2, 1\rceil$	$\lfloor 1, a\rceil$	$\lfloor 1, a^2\rceil$
bi-degree	$(2/3, 2/3)$	$(1/3, 1/3)$	$(1/3, 1/3)$

Table 4.2: Bi-grading of $\mathscr{H}$ for $W = D_4 = x^2y + y^3$

4.5 Bring it together

We now have a multiply graded vector space, $\mathscr{H} = \bigoplus_{g\in G} \mathscr{H}_g$. The state space of each LG construction is the G-invariant subspace of $\mathscr{H}$, $\left(\bigoplus_{g\in G} \mathscr{H}_g\right)^G$. In this section we introduce several ways of determining this space.

As a first approach to determining G-invariants of $\mathscr{H}$, we present a representation theoretic idea. Let V be a $\mathbb{C}G$ module, i.e. V is a vector space with a group action. Define $\pi : V \to V$ by $\pi(v) = \dfrac{1}{|G|}\sum_{g\in G} vg$. This is easily seen to be linear. Also, π is surjective on the invariant subspace V^G since if $v \in V^G$, then $vg = v$ for all $g \in G$ and we have $\pi(v) = \dfrac{1}{|G|}\sum_{g\in G} vg = \dfrac{1}{|G|}\sum_{g\in G} v = v$.

Thus, if we apply π to each basis element of V, we will get a spanning set for V^G. The linearly independent images provide a basis for V^G.

Alternatively, we provide a simplifying description of $\mathscr{H}^G$ here and in the following section. Throughout, we assume G has r conjugacy classes $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_r$ with a fixed choice of representatives $1, g_2, \ldots, g_r$. Also, since conjugacy is transitive, for each conjugacy class $\mathcal{K}_i$ and each $h \in \mathcal{K}_i$ there is a fixed choice of "conjugator" b_h so that $b_h^{-1} g b_h = h$.

First, observe that $\mathscr{H}$ decomposes into a sum over conjugacy classes: $\bigoplus_{g\in G} \mathscr{H}_g = \bigoplus_{i=1}^r \left(\bigoplus_{g\in\mathcal{K}_i} \mathscr{H}_g\right)$. Thus any element of $\mathscr{H}$ decomposes into a sum over conjugacy classes: $\alpha \in \mathscr{H} = \sum_g \lfloor m_g, g\rceil = \sum_{i=1}^r \sum_{g\in\mathcal{K}_i} \lfloor m_g, g\rceil = \sum_{i=1}^r \alpha_i$.

Fact 4.22. *If such an α is G invariant, then the α_i are G invariant.*

Proof. Let $k \in G$. The action of k sends $\lfloor m_g, g\rceil$ to $\lfloor \det(k_{(g)}) m_g \circ k, k^{-1}gk\rceil$; i.e. it moves elements around by conjugacy classes. Since $\alpha = \sum_{i=1}^r \alpha_i$, $m \cdot k = \sum_{i=1}^r m_i \cdot k$. Since the state space is a direct sum of the conjugacy class pieces, $\sum_{i=1}^r m_i = \sum_{i=1}^r m_i \cdot k$ implies $m_i \cdot k = m_i$. □

Lemma 4.23. *If $\alpha_i = \sum_{g\in\mathcal{K}_i} \lfloor m_g, g\rceil$ is nonzero and G-invariant, then $\lfloor m_{g_i}, g_i\rceil \neq 0$.*

Proof. Suppose $\alpha_i = \sum_{g\in\mathcal{K}_i} \lfloor m_g, g\rceil$ is nonzero and G-invariant, but $\lfloor m_{g_i}, g_i\rceil = 0$. There is some $h \in \mathcal{K}_i$ so that $\lfloor m_h, h\rceil \neq 0$. We have a chosen conjugator b_h so that $b_h^{-1} g_i b_h = h$. This means $\lfloor m_{g_i}, g_i\rceil \cdot b_h = \lfloor \det(b_{h(g)}) m_{g_i} \circ b_h, h\rceil$. Since conjugacy is faithful, $k^{b_h} \neq h$ for all $k \in G$. Then $\alpha_i \cdot k = \alpha_i$, and the terms agree by sectors. In particular, $\lfloor m_{g_i}, g_i\rceil \cdot b_h = 0 \cdot b_h = 0 = \lfloor \det(b_{h(g_i)}) m_{g_i} \circ b_h, h\rceil = \lfloor m_h, h\rceil$. This implies $\lfloor m_h, h\rceil = 0$, a contradiction.

□

Proposition 4.24. *If $\sum_{g\in\mathcal{K}_i} \lfloor m_g, g\rceil$ is G-invariant, then*

$$\sum_{g\in\mathcal{K}_i} \lfloor m_g, g\rceil = \sum_{g\in\mathcal{K}_i} \lfloor m_{g_i}, g_i\rceil \cdot b_g.$$

Proof. This statement is sensible because the above lemma guarantees that the g_i term is nonzero if the sum is nonzero. The proposition follows from the faithfulness and transitivity of G on $\mathcal{K}_i$ acting by conjugation. For $h \in \mathcal{K}_i$, consider the action of $b := b_h$ on $\sum_{g\in\mathcal{K}_i} \lfloor m_g, g\rceil$.

We have $\sum_{g\in\mathcal{K}_i}\lfloor \det(b_{(g)})m_g\circ b, b^{-1}gb\rceil = \sum_{g\in\mathcal{K}_i}\lfloor m_g, g\rceil$. Since $\mathscr{H}$ is a direct sum of g-sectors, the terms agree by sectors, so in particular

$$\begin{aligned}\lfloor m_g, b_g\rceil \cdot b &= \lfloor \det(b_{(g)})m_{g_i}\circ b, b^{-1}g_ib\rceil \\ &= \lfloor \det(b_{(g)})m_{g_i}\circ b, h\rceil \\ &= \lfloor m_h, h\rceil.\end{aligned}$$

That is, $\lfloor m_g, g\rceil = \lfloor m_{g_i}, g_i\rceil \cdot b_g$ for all $g\in\mathcal{K}_i$.

□

Example 4.25. In the case of $W = x^2y + y^3$, we calculated the state space in Example 4.21. We find that the invariants are $\{\lfloor 1,1\rceil, \lfloor y^2, 1\rceil, \lfloor 1, a\rceil, \lfloor 1, a^2\rceil\}$.

4.6 $(\bigoplus_{g\in G}\mathscr{H}_g)^G \cong \bigoplus_{\{g\rangle\}}(\mathscr{H}_g)^{C_G(g)}$

The space $(\mathscr{H})^G$ has some redundancy we can use to simplify calculations. Recall from section 4.3.2 where we proved that $(\bigoplus_{g\in G}\mathscr{H}_g)^G$ is a right G-module, that we also established an isomorphism of conjugate g-sectors. We use this isomorphism to condense $\mathscr{H}^G$ to a simpler sum where we only take conjugacy class representatives.

Lemma 4.26. *If $\sum\lfloor m_g, g\rceil$ is G-invariant, then $\sum_{i=1}^r\lfloor m_i, g_i\rceil$ is $C_G(g_i)$-invariant.*

Proof. It suffices to prove this over conjugacy classes: suppose $\sum_{h\in\mathcal{K}_i}\lfloor m_h, h\rceil$ is G-invariant and $k\in C_G(h)$. For any $h\in\mathcal{K}_i$, $\lfloor m_h, h\rceil\cdot k = \lfloor \det(k_{(h)})m_h\circ k, k^{-1}hk\rceil = \lfloor \det(k_{(h)})m_h\circ k, h\rceil$. If $\sum_{h\in\mathcal{K}_i}\lfloor m_h, h\rceil$ is G-invariant, then in particular it is k-invariant and we have $\sum_{h\in\mathcal{K}_i}\lfloor m_h, h\rceil = \sum_{h\in\mathcal{K}_i}\lfloor \det(k_{(h)})m_h\circ k, h\rceil$. Since $k^{-1}ak = k^{-1}bk$ implies $a=b$, the only thing k sends to the h sector is $\lfloor m_h, h\rceil$. This implies $\lfloor \det(k_{(h)})m_h\circ k, h\rceil = \lfloor m_h, h\rceil$, or $\lfloor m_h, h\rceil$ is $C_G(h)$-invariant.

□

Lemma 4.27. *If $\lfloor m, g_i\rceil$ is $C_G(g_i)$-invariant, then $\sum_{a\in\mathcal{K}_i}\lfloor m, g_i\rceil\cdot b_a$ is G-invariant.*

Proof. Suppose $\lfloor m, g_i \rceil$ is $C_G(g_i)$ invariant and $k \in G$. $\left(\sum_{a\in\mathcal{K}_i} \lfloor m, g_i\rceil \cdot b_a\right)\cdot k = \sum_{a\in\mathcal{K}_i} \lfloor m, g_i\rceil \cdot (b_a k)$.

Suppose $(b_a k)^{-1} g_i b_a k = (b_c k)^{-1} g_i b_c k$, meaning two elements have the same action by conjugation on g_i. This gives $a^{-1} g_i a = c^{-1} g_i c$. We imposed earlier a unique choice of conjugator, so this implies $a = c$. This gives us that g permutes conjugators of g_i.

Now we compute $\lfloor m, g_i\rceil \cdot b_a k = \lfloor \det b_a k_{(g_i)}) m \circ b_a k, (b_a k)^{-1} g_i (b_a k)\rceil$. There is some a' so that $(b_a k)^{-1} g_i (b_a k) = a'$. Then $(b_{a'})^{-1} g_i b_{a'} = (b_a k)^{-1} g_i b_a k$, which implies that $b_a k b_{a'}^{-1} \in C_G(g)$. Thus $\lfloor m, g_i\rceil \cdot b_a k b_{a'}^{-1} = \lfloor m, g_i\rceil$, or $\lfloor m, g_i\rceil \cdot b_a k = \lfloor m, g_i\rceil b_{a'}$.

Thus $\sum_{a\in\mathcal{K}_i} \lfloor m, g_i\rceil \cdot (b_a k) = \sum_{a'\in\mathcal{K}_i} \lfloor m, g_i\rceil \cdot b_{a'}$. Since g merely permutes the conjugators, we recover the original sum $\sum_{a\in\mathcal{K}_i} \lfloor m, g_i\rceil \cdot (b_a)$ and conclude that this is G-invariant.

□

Remark. The above fact holds for any $g \in \mathcal{K}_i$ if we adjust our choice of "conjugators."

Putting these facts together, we obtain the following theorem,

Theorem 4.28. *$\mathscr{H}^G \cong \bigoplus_{i=1}^r \mathscr{H}_{g_i}^{C_G(g_i)}$ as vector spaces.*

Proof. The preceding facts let us construct a pair of linear maps between the desired spaces:

$$\varphi : \mathscr{H}^G \to \bigoplus_{i=1}^r \mathscr{H}_{g_i}^{C_G(g_i)},$$
$$\sum_{g\in\mathcal{K}_i} \lfloor m_g, g\rceil \mapsto \lfloor m_{g_i}, g_i\rceil$$
$$\psi : \bigoplus_{i=1}^r \mathscr{H}_{g_i}^{C_G(g_i)} \to \mathscr{H}^G,$$
$$\lfloor m_{g_i}, g_i\rceil \mapsto \sum_{a\in\mathcal{K}_i} \lfloor m, g_i\rceil b_a$$

where these maps are defined on conjugacy class or class representative summands of the respective spaces. The maps are well-defined by lemmas 4.26 and 4.27.

Linearity follows easily, since a linear combination of invariants is again invariant.

It is easy to see that $\varphi\psi$ is the identity on $\bigoplus_{i=1}^{r} \mathscr{H}_{g_i}^{C_G(g_i)}$, since the conjugator of g_i that gives g_i is the identity element. On the other hand, proposition 4.24 guarantees that $\psi\varphi$ is the identity on $\mathscr{H}^G$.

$\square$

4.6.1 Grading on $\bigoplus_{g\in G} \mathscr{H}_g^G$. As indicated in the examples above, invariants aren't always monomials. There is still a well-defined grading on $\mathscr{H}^G$, however. Since the $\mathbb{C}^*$ operator that ultimately defines the degree commutes with every symmetry, we can decompose $\mathscr{H}_{g_i}$ into graded components. The grading on $\bigoplus_{i=1}^{r} \mathscr{H}_{g_i}^{C_G(g_i)}$ is preserved by the isomorphism in Theorem 4.28, so $\mathscr{H}^G$ is appropriately graded.

Chapter 5. Frobenius Algebra

We are ready to develop the pairing and multiplication of $\mathscr{B}_W^G$. While the diagonal $\mathscr{B}$-model has higher level structure (the interested reader can peruse such works as [Web13] for more information), we only extend one more layer beyond the graded vector space to obtain a Frobenius algebra.

Recall that a *Frobenius algebra* is an algebra A over $\mathbb{C}$ with a nondegenerate, symmetric, bilinear pairing that satisfies the Frobenius property. A pairing on A is a function $\langle\cdot,\cdot\rangle : A \times A \to \mathbb{C}$. The Frobenius property is that $\langle ab, c\rangle = \langle a, bc\rangle$ for all $a, b, c \in A$.

In this chapter, we present the pairing and multiplication on $\mathscr{B}_W^G$. Each of these is defined on $\mathscr{H}$, and shown to well-defined on the invariant subspace $\mathscr{B}_W^G = \mathscr{H}^G$. There is a factor, still unknown in the general case, that makes the multiplication associative. We define the multiplication up to this factor, leaving a case-by-case analysis to Chapter 6. We make a few general comments on the identity element of $\mathscr{B}_W^G$, and demonstrate that the multiplication satisfies the Frobenius property.

Remark. Both the $\mathscr{A}_{W,G}$ and $\mathscr{B}_{W,G}$-models have largely the same construction, only differing substantially in the multiplication. Since the $\mathscr{A}$-model multiplication is beyond the scope of this thesis, we say no more about $\mathscr{A}$-model construction.

5.1 Pairing

The first operation on $\mathscr{B}_{W,G}$ we introduce in this chapter is the pairing. First, recall Proposition 4.2, stating that $W|_{V^g}$ is a quasihomogeneous, nondegenerate polynomial with weights $< 1/2$. This allows us to apply the same results as used in the development of the diagonal $\mathscr{A}$- and $\mathscr{B}$- models. These theorems state that $\mathscr{H}_g := \mathscr{Q}_W/JW|_{V^g}$ has a one-dimensional subspace of largest weighed degree spanned by the Hessian determinant, $\det\left(\frac{\partial^2 W|_{V^g}}{\partial x_i \partial x_j}\right)$, of $W|_{V^g}$. This allows us to define the *residue pairing* $\langle \cdot, \cdot \rangle$ on $\mathscr{H}_g$:

Let $\mu_g := \dim \mathscr{H}_g$. For $\lfloor p, g \rceil, \lfloor q, g \rceil \in \mathscr{Q}_{W|_{V^g}}$, the *residue pairing* $\langle \lfloor p, g \rceil, \lfloor q, g \rceil \rangle$ is defined implicitly as the solution of the equation $\lfloor p, g \rceil \lfloor q, g \rceil = \frac{\langle \lfloor p, g \rceil, \lfloor q, g \rceil \rangle}{\mu_g} \text{Hess } W|_{V^g} +$ lower order terms.

Remark. In the above definition, the Hessian Hess $W|_{V^g}$ resulted from a second derivative. This element spans the 1-dimensional space of highest degree. Since $\mathscr{H}_g$ is generated by monomials, Hess $W|_{V^g}$ has a monomial representative. For a given $g \in G_W^{sm}$, we will use Hess g to denote the monic monomial obtained from Hess $W|_{V^g}$; it spans the same space but doesn't have the coefficient. The pairing matrix will keep track of this coefficient.

Since Fix $h =$ Fix h^{-1}, we have $\mathscr{H}_h \cong \mathscr{H}_{h^{-1}}$, and the residue pairing on $\mathscr{Q}_{W|_{\text{Fix } h}}$ induces a pairing

$$\mathscr{H}_h \otimes \mathscr{H}_{h^{-1}} \to \mathbb{C}.$$

The pairing on $\mathscr{H}$ is the direct sum of these pairings. Fixing a basis for $\mathscr{H}$, we denote the pairing by a matrix $\eta_{\alpha,\beta} = \langle \alpha, \beta \rangle$.

Example 5.1. Let $W = x^2y + y^3$. We determined the basis in Example 4.21. Maintaining the order of basis elements as presented there, we determine that the pairing matrix is

Pairing	$\lfloor 1,1 \rceil$	$\lfloor x,1 \rceil$	$\lfloor y,1 \rceil$	$\lfloor y^2,1 \rceil$	$\lfloor 1,a \rceil$	$\lfloor 1,a^2 \rceil$
$\lfloor 1,1 \rceil$				$1/3$		
$\lfloor x,1 \rceil$			$1/3$			
$\lfloor y,1 \rceil$		$1/3$				
$\lfloor y^2,1 \rceil$	$1/3$					
$\lfloor 1,a \rceil$						1
$\lfloor 1,a^2 \rceil$					1	

5.2 $\mathscr{B}$ MULTIPLICATION

With the pairing established, we can define the $\mathscr{B}$-model multiplication. We follow work by Kaufmann and elucidated by Krawitz [Kau02, Kau03, Kau06, Kra10]. The product is defined in terms of "musical isomorphisms" induced by the pairing and a related adjoint operator. Notationally, we use a star: $\lfloor m,g \rceil \star \lfloor n,h \rceil$.

The diagram in figure 5.1 provides the template for the multiplication. In this section we explain how two elements $\lfloor m,g \rceil$ and $\lfloor n,h \rceil$ are projected into $\mathscr{H}_{g\cap h}$, multiplied together and sent around the diagram of figure 5.1.

$$\begin{tikzcd}
\mathscr{H}_g \arrow[dr, "\kappa_g"] & & \\
& \mathscr{H}_{g\cap h} \arrow[dd, "\eta^\flat"] & \mathscr{H}_{gh} \arrow[l, two heads, "\kappa_{gh}"'] \\
\mathscr{H}_h \arrow[ur, "\kappa_h"'] & & \\
& \widehat{\mathscr{H}_{g\cap h}} \arrow[r, "\widehat{\kappa_{gh}}"'] & \widehat{\mathscr{H}_{gh}} \arrow[uu, "\eta^\sharp"']
\end{tikzcd}$$

Figure 5.1: The multiplication diagram (again)

5.2.1 Verifying the Multiplication Diagram. In this section, we justify the diagram in Figure 5.1. We don't walk through the multiplication process; we only verify that each object and map in the diagram exist and satisfies the necessary properties. In the process, we show that there is an equivalent diagram for $\mathscr{H}'_g$ and $\mathscr{H}'_h$. In the following section we will walk through the actual operation of the multiplication.

An element in $\mathscr{H}_g$ and an element in $\mathscr{H}_h$ should multiply to give an element in $\mathscr{H}_{gh}$. The first step in defining this product is to project elements of $\mathscr{H}_g$ and $\mathscr{H}_h$ into a common ring. The ring in common is the Milnor ring $\mathscr{H}_{g\cap h} := \mathscr{Q}_{W|_{V^g\cap V^h}}$ obtained by restricting to the fixed locus of both g and h, $V^g \cap V^h$. To simplify notation, we will write $V^g \cap V^h$ as $V^{g\cap h}$. Also, We will write $W|_{V^g\cap V^h}$ as $W|_{V^{g\cap h}}$ or even $W_{g\cap h}$.

We will shortly demonstrate that an equivalent expression of $\mathscr{H}_g$ is $\mathscr{H}'_{g\cap h} := \mathscr{Q}_W/\lfloor I_g + I_h, JW\rceil$.

Remark. Similar to the reasoning in Lemmas 4.8 and 4.9, $\mathscr{H}_{g\cap h}$ and $\mathscr{H}'_{g\cap h}$ are coordinate independent. That is, for any two bases $\{v_1,\dots,v_n\},\{w_1,\dots,w_n\}$ for V, with corresonding dual bases $\{x_1,\dots,x_n\}$ and $\{y_1,\dots,y_n\}$, the spaces $\mathbb{C}[x_1,\dots,x_n]/(JW_x + I_{g,x} + I_{h,x})$ and $\mathbb{C}[\mathbf{y}]/(JW_y + I_{g,y} + I_{h,y})$ are isomorphic.

Similarly, for any two bases $\{v_1,\dots,v_k\},\{w_1,\dots,w_k\}$ for $V^g\cap V^h$, with corresonding dual bases $\{x_1,\dots,x_k\}$ and $\{y_1,\dots,y_k\}$, the spaces $\mathbb{C}[x_1,\dots,x_k]/JW|_{V^g_v}$ and $\mathbb{C}[\mathbf{y}]/JW|_{V^g_w}$ are isomorphic.

Proposition 5.2. $\mathscr{H}'_{g\cap h} \cong \mathscr{H}_{g\cap h}$

Proof. This proof is similar to the proof Theorem 4.10. If $\iota_{g\cap h} : V^g\cap V^h \hookrightarrow V$ is the inclusion map, then the adjoint $\widehat{\iota_{h,g}} : \widehat{V} \to \widehat{V^g\cap V^h}$ extends to an isomorphism $\mathbb{C}[\widehat{V}] \to \mathbb{C}[\widehat{V^g\cap V^h}]$. As in Theorem 4.10, the desired isomorphism between $\mathscr{H}'_{g\cap h}$ and $\mathscr{H}_{g\cap h}$ is the filler in the diagram of figure 5.2. The κ maps of the diagram are the obvious quotient maps. The filler exists if $\widehat{\iota_{g\cap h}}(JW+I_g+I_h) \subset JW|_{V^{g\cap h}}$ and is an isomorphism if $\widehat{\iota_{g\cap h}}(JW+I_g+I_h) = JW|_{V^{g\cap h}}$.

$$\begin{tikzcd}
\mathbb{C}[\widehat{V}] \arrow[r, two heads, "\widehat{\iota_{g\cap h}}"] \arrow[d, two heads, "\kappa'_{g,h}"'] & \mathbb{C}[\widehat{V^{g\cap h}}] \arrow[d, two heads, "\kappa_{g,h}"] \\
\mathbb{C}[\widehat{V}]/(JW + I_g + I_h) \arrow[r, dashed, two heads] & \mathbb{C}[\widehat{V^{g\cap h}}]/JW_{g\cap h}
\end{tikzcd}$$

Figure 5.2: The diagram of the isomorphism between $\mathscr{H}_{g\cap h}$ and $\mathscr{H}'_{g\cap h}$.

Since $\mathscr{H}_g$ and $\mathscr{H}'_g$ are coordinate independent, we only show the isomorphism in a special coordinate frame. Let $\{e_1,\dots,e_r\}$ be a basis for $V^g\cap V^h$. Clearly, both g and h are diagonal (in fact, act as the identity) on this basis. Extend to a basis $\{e_1, \ldots, e_k\}$ for V^g and again to a basis $\{e_1, \ldots, e_n\}$ on which g acts diagonally. Let $\{s_1, \ldots, s_n\}$ be the canonical dual basis to $\{e_1, \ldots, e_n\}$. Note that, while h doesn't necessarily act diagonally on the basis elements $e_{r+1}, \ldots, e_n$, all eigenvectors of h with eigenvalue other than 1 are in the span of $\{e_{r+1}, \ldots, e_n\}$. This indicates that the eigenvectors of $\widehat{h}$ with eigenvalue other than 1 are in the span of $\{s_{r+1}, \ldots, s_n\}$.

First we'll show that $\widehat{\iota_{g\cap h}}(I_g + I_h) \subset JW|_{V^{g\cap h}}$. Note that $\widehat{\iota}(s_i) = \begin{cases} s_i & i \leq r \\ 0 & i > r \end{cases}$. Let $m \in I_g$, so m has a factor s_j that is not fixed by g. This means $j > r$, so $\widehat{\iota}(m) = 0 \in JW_g$. Now let $m \in I_h$. By the remark at the end of the preceding paragraph, m has a factor in the span of $\{s_{r+1}, \ldots, s_n\}$ and so $\widehat{\iota}(m) = 0$.

Now we show that $\widehat{\iota_{g\cap h}}(JW) \subset JW|_{V^{g\cap h}}$. Since JW is generated by derivatives of W, it is sufficient to look at derivatives of monomials of W. We consider several cases of $\widehat{\iota}(\frac{\partial m}{\partial s_i})$ for m a monomial of W.

- If m has no factor of s_i, then $\widehat{\iota}(\frac{\partial m}{\partial s_i})$ is 0.
- If m has any factor s_j where $j \neq i$ and $j > r$, then $\widehat{\iota}(\frac{\partial m}{\partial s_i}) = 0$.

For the other cases, suppose all factors of m are fixed by g and h, besides possibly s_i.

- The variable s_i is fixed by g and h, or $i \leq r$. In this case, m has no factors s_j for $j \geq r$. Then m is a monomial of $W|_{V^g \cap V^h} = \widehat{\iota}(W)$, so $\widehat{\iota}(\frac{\partial m}{\partial s_i}) \in JW|_{g\cap h}$.
- The variable s_i is not fixed by g, so $r > k$. Then the power of s_i is greater than 1. *Proof* Suppose otherwise: Since g is a symmetry of W, $W \circ g = W$ and so $m \circ g = m$. This means $\frac{\partial m\circ g}{\partial s_i} = \lambda_i \frac{\partial m}{\partial s_i}$ and $\frac{\partial m\circ g}{\partial s_i} = \frac{\partial m}{\partial s_i}$. Then $\lambda_i = 1$, a contradiction.

  Since m has a factor of s_i with power greater than 1, $m \in I_{g\cap h}$ and $\widehat{\iota}(\frac{\partial m}{\partial s_i}) = 0$.
- The variable s_i is fixed by g but not h, so $r < i \leq k$. This case is similar to the previous one. First we show that the power of s_i is greater than 1.

Suppose, by way of contradiction, that $m = s_1^{a_1} \dots s_r^{a_r} s_i$. By coordinate independence, we can switch bases to an extension of $\{e_1, \dots, e_r\}$ that diagonalizes h. We consider all "m-like" monomials of W, meaning monomials of the form $s_1^{a_1} \dots s_r^{a_r} s_j$ for $j \geq r$. The coordinate switch sends $s_1^{a_1} \dots s_r^{a_r} s_j$ to $s_1^{a_1} \dots s_r^{a_r} (\sum_{k=r+1}^{n} s'_k)$, i.e. it takes the sum of m-like monomials to m-like monomials in the new coordinates. Not all of the image monomials are 0 or the coordinate transformation would have determinant 0. Since h acts diagonally on these coordinates, we have the previous case and the previous conclusion for each m-like monomial in the new coordinates.

This means there is no m-like monomial in the new coordinates, so there is no m or the power of s_i is greater than 1. Thus $\widehat{\iota}(\frac{\partial m}{\partial s_i}) = 0$.

In every case, $\widehat{\iota}(\frac{\partial m}{\partial s_i})$ is an element of $JW_{g \cap h}$ for m a monomial of W. This gives us $\widehat{\iota_{g \cap h}}(JW + I_g + I_h) \subset JW|_{V^{g \cap h}}$.

Now we show that $\widehat{\iota_{g \cap h}}^{-1}(JW|_{V^{g \cap h}}) \subset JW + I_g + I_h$. Suppose $m \in \widehat{\iota_{g \cap h}}^{-1}(JW|_{V^{g \cap h}})$. If $\widehat{\iota}(m) = 0$, then m has a factor s_j for some $j > r$. If $j > k$, then $m \in I_g$. If $r < j \leq k$, then s_j is fixed by $\widehat{g}$ but not in the basis for $\widehat{V^g \cap V^h}$. This means $s_j \in I_h$. Now suppose $\widehat{\iota_{g \cap h}}(m) \neq 0$. Then $\widehat{\iota_{g \cap h}}(m) = m \in JW_g \subset JW$, and so we have that $\widehat{\iota_{g \cap h}}^{-1}(JW|_{V^{g \cap h}})$ is a subset of $JW + I_g + I_h$.

Thus $\widehat{\iota_{g \cap h}}(JW + I_g + I_h) = JW|_{V^{g \cap h}}$ and so $\mathscr{H}_{g \cap h} \cong \mathscr{H}'_{g \cap h}$.

$\square$

By Theorem 4.5, $\mathscr{H}_g = \mathscr{Q}_W / \lfloor I_g, JW \rceil$ and

$$\mathscr{H}_{g \cap h} = \mathscr{Q}_W / \lfloor I_g + I_h, JW \rceil = \left(\mathscr{Q}_W / \lfloor I_g, JW \rceil \right) / \lfloor I_h, JW + I_g \rceil.$$

Clearly $\mathscr{H}_{g \cap h}$ is a quotient of $\mathscr{H}_g$, so we have a natural quotient map $\kappa_g : \mathscr{H}_g \to \mathscr{H}_{g \cap h}$, which is well-defined and surjective. Similarly, there is a surjective map κ_h from $\mathscr{H}'_h \to \mathscr{H}'_{g \cap h}$. Using the isomorphism of Proposition 5.2, we have surjective maps (which we also label κ_g and κ_h) from $\mathscr{H}_g$ and $\mathscr{H}_h$ to $\mathscr{H}_{g \cap h}$.

Lemma 5.3. *There is also a well-defined, surjective map* κ_{gh} *from* $\mathscr{H}'_{gh}$ *to* $\mathscr{H}'_{g\cap h}$.

Proof. Here, the spaces are $\mathscr{H}'_{gh} = \mathscr{Q}_W/\lfloor I_{gh}, JW\rceil$ and $\mathscr{H}'_{g\cap h} = \mathscr{Q}_W/\lfloor I_g + I_h, JW\rceil$. There is a quotient map from $\mathscr{Q}_W \to \mathscr{Q}_W/\lfloor I_g + I_h\rceil$; this descends to a well-defined map $\mathscr{H}'_{gh} \to \mathscr{H}'_{g\cap h}$ if $\lfloor I_{gh}\rceil \subset \lfloor I_g + I_h\rceil$.

To that end, recall from Section 4.2 that I_{gh} is generated by elements of the form $y - \widehat{gh}(y)$. We can add and subtract $\widehat{g}(y)$ to get $y - \widehat{g}(y) + \widehat{g}(y) - \widehat{g}(\widehat{h}(y)) = y - \widehat{g}(y) + \widehat{g}(y - \widehat{h}(y))$. Now add and subtract $y - \widehat{h}(y)$ to get

$$\begin{aligned} y - \widehat{g}(y) + y - \widehat{h}(y) - \left(y - \widehat{h}(y) - \widehat{g}(y - \widehat{h}(y)\right) \\ = (y - \widehat{g}(y)) - (\alpha - \widehat{g}(\alpha)) + \left(y - \widehat{h}(y)\right), \end{aligned}$$

where we set $\alpha := y - \widehat{h}(y)$. This is clearly in $I_g + I_h$.

$\square$

Remark. Since $\mathscr{H}_{gh} \cong \mathscr{H}'_{gh}$ and $\mathscr{H}_{g\cap h} \cong \mathscr{H}'_{g\cap h}$, we get a surjective map (which we also label κ_{gh}) from $\mathscr{H}_{gh}$ to $\mathscr{H}_{g\cap h}$. Also, by Proposition 4.1, the adjoint map $\widehat{\kappa_{gh}} : \widehat{\mathscr{H}_{g\cap h}} \to \widehat{\mathscr{H}_{gh}}$ is injective.

We now discuss the remaining maps from the diagram in figure 5.1, the maps named $\eta^\sharp$ and $\eta^\flat$. These arise because we assume a nondegenerate pairing on $\mathscr{H}_{g\cap h}$ and $\mathscr{H}_g$. The pairing on $\mathscr{H}_{g\cap h}$ lets us construct an isomorphism $\eta^\flat : \mathscr{H}_{g\cap h} \to \widehat{\mathscr{H}_{g\cap h}}$ between $\mathscr{H}_{g\cap h}$ and its dual. Also, we have an isomorphism $\eta^\sharp : \widehat{\mathscr{H}_{gh}} \to \mathscr{H}_{gh}$. This is similar to the map between a Hilbert space and its dual (cf. chapter 11 of [DF04]). The pairing is not an inner product, but that it is nondegenerate is sufficient. The procedure is common in Riemannian geometry, and proofs of this fact can be found online or in many texts on differential or Riemannian geometry. We merely state here this poposition and refer the reader to other sources for the proof.

Proposition 5.4. *Let* V *be a finite-dimensional vector space over a field* F *with a nondegenerate, symmetric, bilinear form* $\eta : V \times V \to F$. *Then* $V \cong \widehat{V}$.

5.2.2 The Product. Now that we have verified the diagram in figure 5.1, we use it to define a multiplication on $\mathscr{H}$.

- The first step in defining $\lfloor m, g\rceil \star \lfloor n, h\rceil$ is to project both $\lfloor m, g\rceil$ and $\lfloor n, h\rceil$ into $\mathscr{H}_{g\cap h}$ via the surjective maps of the diagram in 5.1.

- Once there, multiply by a factor $\epsilon_{g,h} \in \mathscr{H}_{g\cap h}$. This factor makes the multiplication associative and respect the bi-grading. In the diagonal case,

$$\epsilon_{g,h} = \begin{cases} 1, & V^g + V^h + V^{gh} = V \\ 0, & \text{otherwise} \end{cases} .$$

  This condition does not work in the nondiagonal case, as we will see later. We do not yet have a general formula for the $\epsilon_{g,h}$ factor, but we demonstrate how it is determined case by case as we compute several examples. A significant part of each example is showing what $\epsilon_{g,h}$ must be in each case in order to preserve the bi-gradings and be associative.

  To simplify our work, we note that we need only check $\lfloor 1, g\rceil \star (\lfloor 1, h\rceil \star \lfloor 1, k\rceil) = (\lfloor 1, g\rceil \star \lfloor 1, h\rceil) \star \lfloor 1, k\rceil$. This follows since each ring $\mathscr{H}_g$ is actually a $\mathscr{Q}_W$ or $\mathscr{H}_1$ module (by the projection of $\mathscr{Q}_W$ into $\mathscr{H}_g$) and the product of the identity sector is associative with respect to other multiplications.

- The next step in the multiplication is to take the product $\pi := \kappa_g(\lfloor m, g\rceil)\, \kappa_h(\lfloor n, h\rceil)\, \epsilon_{g,h}$ through the maps $\eta^\flat$, $\widehat{\kappa_{gh}}$, and $\eta^\sharp$. These maps are well-defined, linear, and injective, so we have a well-defined product:

$$\lfloor m, g\rceil \star \lfloor n, h\rceil := \eta^\sharp \left(\widehat{\kappa_{gh}} \left(\eta^\flat(\pi)\right)\right),$$
$$\pi := \kappa_g(\lfloor m, g\rceil)\, \kappa_h(\lfloor n, h\rceil)\, \epsilon_{g,h}.$$

  Note that the product is determined by the image of $\epsilon_{g,h}$ under $\eta^\sharp \circ \widehat{\kappa_{gh}} \circ \eta^\flat$.

This gives a well-defined binary operation on $\mathscr{H}$. It is easily seen to distribute with addition. Shortly we demonstrate the existence of an identity. We show that the construction satisfies the Frobenius property. In the next chapter we examine several examples and verify associativity and that the multiplication respects the grading of $\mathscr{H}$.

Example 5.5. One way to understand the map into $\widehat{\mathscr{H}_{gh}}$ is to consider row vectors. The function $\eta^\flat : \mathscr{H}_{g\cap h} \to \widehat{\mathscr{H}_{g\cap h}}$ sends an element $m \in \mathscr{H}_{g\cap h}$ to a row vector where each entry is the value $\langle m, b\rangle$ for each basis element b of $\mathscr{H}_{g\cap h}$. The row vectors of the pairing matrix correspond to basis elements of $\mathscr{H}_{gh}$ under $\eta^\sharp$ and span $\widehat{\mathscr{H}_{gh}}$. Any appropriate sized row vector is a function in $\widehat{\mathscr{H}_{gh}}$, and will be a linear combination of row vectors of the pairing matrix. This linear combination indicates the image of the row vector under $\eta^\sharp$.

Consider P_8 and $\lfloor 1, (12)\rceil \star \lfloor 1, (13)\rceil$. First we project both into $\mathscr{H}_{g\cap h}$, which looks like $\mathscr{H}_{(123)} = \{\lfloor 1, (123)\rceil, \lfloor s, (123)\rceil\}$. We get $\lfloor 1, g\cap h\rceil \star \lfloor 1, g\cap h\rceil = \lfloor 1, g\cap h\rceil$. In this case, $\epsilon_{g,h} = 1$. Mapping into the dual and over, we get the function $\langle \lfloor 1, g\cap h\rceil, \lfloor \cdot, g\cap h\rceil\rangle$ in $\widehat{\mathscr{H}_{(132)}}$. We look at what this does to the basis elements $\lfloor 1, (132)\rceil$ and $\lfloor t, (132)\rceil$.

$\langle \lfloor 1, g\cap h\rceil, \lfloor 1, g\cap h\rceil\rangle = 0$. $\langle \lfloor 1, g\cap h\rceil, \lfloor s, g\cap h\rceil\rangle = 1/9$. So $\widehat{f}(\eta^\flat(\lfloor 1, g\cap h\rceil)) = [0 \quad 1/9]$, understood as the values of the function on the basis elements. This is the first row vector of the pairing matrix, corresponding to $\lfloor 1, (132)\rceil$. So $\lfloor 1, (12)\rceil \star \lfloor 1, (13)\rceil = \lfloor 1, (132)\rceil$.

5.2.3 Check on the identity sector. Let e be the identity element of G. We assert that $\lfloor 1, e\rceil$ is the identity element of the multiplication. First, note that multiplication with anything from the identity sector greatly reduces the diagram for the multiplication, since $V^g \cap V^e = V^g \cap V = V^g$ and $ge = g$. The product of $\lfloor m, e\rceil$ and $\lfloor n, g\rceil$ is the element $\lfloor mn, g\rceil$. Note that, since e is in the center of G and multiplication of polynomials is commutative, $\lfloor m, e\rceil \star \lfloor n, g\rceil = \lfloor n, g\rceil \star \lfloor m, g\rceil$. Moreover, for any element $h \in Z(G)$, $\lfloor m, g\rceil \star \lfloor n, h\rceil = \lfloor n, h\rceil \star \lfloor m, g\rceil$. In particular, $\lfloor 1, e\rceil \star \lfloor n, g\rceil = \lfloor n, g\rceil = \lfloor n, g\rceil \star \lfloor 1, e\rceil$.

5.2.4 The image of $\lfloor$Hess $g\cap h, g\cap h\rceil$. Here we discuss what happens if two elements project and multiply to Hess $g\cap h$ in $\mathscr{H}_{g\cap h}$. This is a special product that arises often, so we discuss it here in general. We recall the template for our multiplication:

$$\begin{tikzcd} \mathscr{H}_{g\cap h} \arrow[d, "\eta^\flat"] & \mathscr{H}_{gh} \arrow[l, two heads, "f"'] \\ \widehat{\mathscr{H}}_{g\cap h} \arrow[r, "\widehat{f}"'] & \widehat{\mathscr{H}}_{gh} \arrow[u, "\eta^\sharp"'] \end{tikzcd}$$

We track it around the diagram. Under $\widehat{f}\circ\eta^\flat$, $\lfloor \text{Hess } g\cap h, g\cap h\rceil$ becomes the linear functional

$$\begin{aligned} \lfloor m,(gh)^{-1}\rceil &\mapsto \langle \lfloor \text{Hess } g\cap h, g\cap h\rceil, \lfloor m, g\cap h\rceil\rangle \\ &= \begin{cases} \langle \lfloor \text{Hess } g\cap h, g\cap h\rceil, \lfloor 1, g\cap h\rceil\rangle & \lfloor m, g\cap h\rceil = \lfloor 1, g\cap h\rceil \\ 0 & \text{else} \end{cases} \\ &= \eta^\flat\left(\frac{\langle \lfloor \text{Hess } g\cap h, g\cap h\rceil, \lfloor 1, g\cap h\rceil\rangle}{\langle \lfloor 1, gh\rceil, \lfloor \text{Hess } (gh)^{-1},(gh)^{-1}\rceil\rangle}\lfloor \text{Hess } (gh)^{-1},(gh)^{-1}\rceil\right). \end{aligned}$$

Under $\eta^\sharp$, this maps to $\dfrac{\langle \lfloor \text{Hess } g\cap h, g\cap h\rceil, \lfloor 1, g\cap h\rceil\rangle}{\langle \lfloor 1, gh\rceil, \lfloor \text{Hess } (gh)^{-1},(gh)^{-1}\rceil\rangle}\lfloor \text{Hess } (gh)^{-1},(gh)^{-1}\rceil$.

Note that in the case when $V^g\cap V^h$ is trivial, $\lfloor \text{Hess } g\cap h, g\cap h\rceil = \lfloor 1, g\cap h\rceil$ and $\langle \lfloor \text{Hess } g\cap h, g\cap h\rceil, \lfloor 1, g\cap h\rceil\rangle = 1$. Thus $\lfloor 1, g\rceil \star \lfloor 1, h\rceil = \dfrac{1}{\langle \lfloor 1, gh\rceil, \lfloor \text{Hess } (gh)^{-1},(gh)^{-1}\rceil\rangle}\lfloor \text{Hess } gh, gh\rceil$ when $V^g\cap V^h = \{0\}$.

5.3 Frobenius property of the pairing

The process for the multiplication, odd as it looks, is designed to satisfy the Frobenius property. To see this, consider three elements $\lfloor a, g\rceil$, $\lfloor b, h\rceil$, and $\lfloor c, k\rceil$. We wish to show that

$$\langle \lfloor a, g\rceil \star \lfloor b, h\rceil, \lfloor c, k\rceil\rangle = \langle \lfloor a, g\rceil, \lfloor b, h\rceil \star \lfloor c, k\rceil\rangle.$$

First, recall that the pairing is defined on inverse sectors. That is, $\langle \lfloor a, g\rceil \star \lfloor b, h\rceil, \lfloor c, k\rceil\rangle = \langle \lfloor a, g\rceil, \lfloor b, h\rceil \star \lfloor c, k\rceil\rangle = 0$ unless $gh = k^{-1}$ (the same as $g^{-1} = hk$. WLOG, say $k = (gh)^{-1}$.

Recall the multiplication template

$$\begin{tikzcd}
\mathscr{H}_g \arrow[dr] & & \\
& \mathscr{H}_{g\cap h} \arrow[dd, "\eta^\flat"] & \mathscr{H}_{gh} \arrow[l, two heads, "f"'] \\
\mathscr{H}_h \arrow[ur] & & \\
& \widehat{\mathscr{H}_{g\cap h}} \arrow[r] & \widehat{\mathscr{H}_{gh}} \arrow[uu, "\eta^\sharp"']
\end{tikzcd}$$

The product of $\lfloor a, g\rceil$ and $\lfloor b, c\rceil$ becomes $\lfloor ab\epsilon_{g,h}, g \cap h\rceil$ in $\mathscr{H}_{g\cap h}$. Tracing through the diagram to $\widehat{\mathscr{H}_{g\cap h}}$, we get the map $\langle ab\epsilon_{g,h}, f(\cdot)\rangle_{\mathscr{H}_{g\cap h}}$ on $\mathscr{H}_{gh}$. This corresponds to $\eta^\sharp(\lfloor a, g\rceil \star \lfloor b, h\rceil) = \langle\lfloor a, g\rceil \star \lfloor b, h\rceil, \cdot\rangle_{gh}$. Then $\langle\lfloor a, g\rceil \star \lfloor b, h\rceil, \lfloor c, k\rceil\rangle_{gh} = \langle ab\epsilon_{g,h}, c\rangle_{g\cap h}$. This pairing is 0 unless

$$abc\epsilon_{g,h} = \langle ab\epsilon_{g,h}, c\rangle_{g\cap h} \frac{\text{Hess } W|_{V^{g\cap h}}}{/\dim(\mathscr{H}_{g\cap h})}.$$

On the other hand, a similar diagram for the product $\lfloor b, h\rceil \star \lfloor c, (gh)^{-1}\rceil$ has $\langle\lfloor a, g\rceil, \lfloor b, h\rceil \star \lfloor c, (gh)^{-1}\rceil = \langle a, \epsilon_{h,(gh)^{-1}} bc\rangle$. This is zero unless

$$abc\epsilon_{h,(gh)^{-1}} = \langle a, \epsilon_{h,(gh)^{-1}} bc\rangle_{h,(gh)^{-1}} \frac{\text{Hess } W|_{V^{h,(gh)^{-1}}}}{\dim(\mathscr{H}_{h,(gh)^{-1}})}.$$

Now observe that these quantities are equal: $V^{(gh)^{-1}} = V^{gh}$, so $V^h \cap V^{(gh)^{-1}} = V^h \cap V^{gh} = V^h \cap V^g$. Then $\mathscr{H}_{g\cap h} = \mathscr{H}_{h,(gh)^{-1}}$. If we impose the modest condition that $\epsilon_{g,h} = \epsilon h, (gh)^{-1}$, then

$$\begin{aligned}
\langle\lfloor a, g\rceil \star \lfloor b, h\rceil, \lfloor c, k\rceil\rangle_{gh} &= \langle ab\epsilon_{g,h}, c\rangle_{g\cap h} \\
&= \langle ab\epsilon_{g,h} c\rangle_{g\cap h} \\
&= abc\epsilon_{g,h} * \frac{\text{Hess } W|_{V^{g\cap h}}}{\dim(\mathscr{H}_{g\cap h})} \\
&= abc\epsilon_{h,(gh)^{-1}} * \frac{\text{Hess } W|_{V^{h,(gh)^{-1}}}}{\dim(\mathscr{H}_{h,(gh)^{-1}})} \\
&= \langle\lfloor a, g\rceil, \lfloor b, h\rceil \star \lfloor c, k\rceil\rangle_{gh}
\end{aligned}$$

6.1 $x^2y + y^3$

We determined the state space in Example 4.21. For reference, and to introduce a convenient numbering, we present the basis again here:

label	**1**	B2	B3
basis element	$\lfloor 1, 1 \rceil$	$\lfloor x, 1 \rceil$	$\lfloor y, 1 \rceil$
bi-degree	(0,0)	$(1/3, 1/3)$	$(1/3, 1/3)$
label	B4	B5	B6
basis element	$\lfloor y, 1 \rceil$	$\lfloor 1, a \rceil$	$\lfloor 1, a^2 \rceil$
bi-degree	$(2/3, 2/3)$	$(1/3, 1/3)$	$(1/3, 1/3)$

Table 6.1: A basis for $\mathscr{B}_W^G$ when $W = x^2y + y^3$ and $G = G_W^{sm} \cap SL(2, \mathbb{C})$

It remains to calculate the multiplication table and verify associativity and the Frobenius property. The multiplication in $\mathscr{H}_1$ is straightforward and will be presented without calculation. As for products involving $\lfloor 1, a \rceil$ or $\lfloor 1, a^2 \rceil$, we must be more careful. We demonstrated in 5.2 that **1** acts as the identity. For products of the form $\lfloor m \neq 1, 1 \rceil \star \lfloor 1, g \rceil$ where $g = a, a^2$, these must be 0 since $\lfloor m \neq 1, 1 \rceil$ projects to 0 in $\mathscr{H}_g$.

Now consider $\lfloor 1, g \rceil \star \lfloor 1, g \rceil$. The product, following the diagram of 5.2, is $\lfloor 1, g^2 \rceil$. Note that the sum of the degrees is (2/3,2/3), but the degree of $\lfloor 1, g^2 \rceil$ is 1/3. In this example, it suffices to keep the same condition in the diagonal case for $\epsilon_{g,h}$ (see Chapter 2 and Section 5.2); namely, that $\epsilon_{g,h}$ be 0 unless $V^g + V^g + V^{gg} = V$, and 1 in that case. Since $V^g + V^g + V^{gg} = \{0\}$, we determine that $\lfloor 1, g \rceil \star \lfloor 1, g \rceil = 0$.

Recall also from 5.2 that we determined $\lfloor 1, g \rceil \star \lfloor 1, g^2 \rceil = 12\lfloor y^2, 1 \rceil = 12B4$. This lets us present the complete the multiplication table in Table 6.1. In this table the blanks indicate zeros.

It is easy to check that the multiplication respects the grading. Since the definitions coincide with the diagonal case, the multiplication on the identity sector is already known to be associative. A simple check verifies associativity on the a and a^2 sectors.

$\star$	**1**	B2	B3	B4	B5	B6
1	**1**	B2	B3	B4	B5	B6
B2	B2	B4				
B3	B3		B4			
B4	B4					
B5	B5					12 B4
B6	B6				12 B4	

Table 6.2: The multiplication table for $\mathscr{B}_W^G$ when $W = x^2y + y^3$ and $G = G_W^{sm} \cap SL(2, \mathbb{C})$

6.1.1 Invariants. The invariants are **1**, B4, B5, and B6. The multiplication is easily seen to be closed with respect to invariants, so $\mathscr{B}_W^G$ is a Frobenius Algebra.

6.2 $x^3y + y^4$

Let $W = x^3y + y^4$ and ω be a third root of unity. The first step in determining $\mathscr{B}_W^G$ is choosing a group; we will use G_W^{sm}. Direct computation shows that

$$\begin{aligned} G^{sm} \cap SL \quad = \quad & \left\{ 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; -1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \; i = \frac{1}{3}\begin{pmatrix} -2\omega - 1 & 4\omega + 2 \\ 2\omega + 1 & 2\omega + 1 \end{pmatrix}, \right. \\ & -i = \frac{1}{3}\begin{pmatrix} 2\omega + 1 & -4\omega - 2 \\ -2\omega - 1 & -2\omega - 1 \end{pmatrix}, \; j = \frac{1}{3}\begin{pmatrix} -2\omega - 1 & -2\omega - 4 \\ \omega + 1 & 2\omega + 1 \end{pmatrix}, \\ & -j = \frac{1}{3}\begin{pmatrix} 2\omega + 1 & 2\omega + 4 \\ \omega - 1 & -2\omega - 1 \end{pmatrix}, \; ij = \frac{1}{3}\begin{pmatrix} 2\omega + 1 & 2\omega - 2 \\ \omega + 2 & -2\omega - 1 \end{pmatrix}, \\ & \left. -ij = \begin{pmatrix} -2\omega - 1 & -2\omega + 2 \\ -\omega - 2 & 2\omega + 1 \end{pmatrix} \right\} \end{aligned}$$

This group is isomorphic to the quaternion group Q_8, with the above elements, as listed, corresponding to the elements $1, -1, i, i^3, j, j^3, ij, (ij)^3$. All of the above elements have trivial fixed locus but the identity.

Proceeding, we compute the state space. First, $\mathscr{H}_1$ is the Milnor ring of W, the space

number	**1**	B2	B3	B4	B5	B6
basis element	$\lfloor 1,1 \rceil$	$\lfloor x,1 \rceil$	$\lfloor y,1 \rceil$	$\lfloor x^2,1 \rceil$	$\lfloor xy,1 \rceil$	$\lfloor y^2,1 \rceil$
bi-degree	(0,0)	(1/4,1/4)	(1/4,1/4)	(1/2,1/2)	(1/2,1/2)	(1/2,1/2)
number	B7	B8	B9	B10	B11	B12
basis element	$\lfloor xy^2,1 \rceil$	$\lfloor y^3,1 \rceil$	$\lfloor xy^3,1 \rceil$	$\lfloor 1,-1 \rceil$	$\lfloor 1,i \rceil$	$\lfloor 1,-i \rceil$
bi-degree	(3/4,3/4)	(3/4,3/4)	(1,1)	(1/2,1/2)	(1/2,1/2)	(1/2,1/2)
number	B13	B14	B15	B16		
basis element	$\lfloor 1,j \rceil$	$\lfloor 1,-j \rceil$	$\lfloor 1,ij \rceil$	$\lfloor 1,-ij \rceil$		
bi-degree	(1/2,1/2)	(1/2,1/2)	(1/2,1/2)	(1/2,1/2)		

Table 6.3: Basis elements of $\mathscr{B}_W^G$ when $W = x^3y + y^4$ and $G = G_W^{sm} \cap SL(2, \mathbb{C})$

$\mathrm{Span}_C\{1, x, y, x^2, xy, y^2, xy^2, y^3, xy^3\}$. Now, $\mathscr{H} = \mathscr{H}_1 \oplus_{g\neq 1} \mathrm{Span}_{\mathbb{C}}\{\lfloor 1, g\rceil\}$. For the grading, $\deg(\lfloor x^a y^b, 1\rceil) = (\frac{a+b+2}{4}, \frac{a+b+2}{4}) + (0,0) - (\frac{2}{4}, \frac{2}{4}) = (\frac{a+b}{4}, \frac{a+b}{4})$, where $a = 0, 1$ and $b = 1, 2, 3$ or $a = 2, b = 0$.

The group element -1 has eigenvalues -1, -1, so the age is $1/2 + 1/2 = 1$. $\deg(\lfloor 1, -1\rceil) = (0, 0) + (1, 1) - (1/2, 1/2) = (1/2, 1/2)$. All other group elements have eigenvalues $\pm i$, so each has age $1/4 + 3/4 = 1$. $\deg(\lfloor 1, g\rceil_{g\neq 1,-1}) = (1/2, 1/2)$. To summarize, Table 6.2 lists the basis elements of $\mathscr{H}_W$ with their bi-degrees and a reference number for later tables.

The pairing matrix block on $\mathscr{H}_1$ is given by table 6.2. All other pairings for elements of $\mathscr{H}_1$ are zero.

	1	B2		B8	B9
1					1/12
B2				1/12	
			$\vdots$		
B8		1/12			
B9	1/12				

Table 6.4: The pairing on $\mathscr{H}_1$ for $W = x^3y + y^4$.

For $\alpha = \lfloor 1, g \neq 1\rceil$, the pairing of α with any other β is given by the following function:

$$\langle \alpha, \beta \rangle = \begin{cases} 1, & \beta = \lfloor 1, g^{-1} \rceil \\ 0 & \text{else} \end{cases}$$

The multiplication on the $\mathscr{H}_1$ block, as in the diagonal case, is

$\star$	**1**	B2	B3	B4	B5	B6	B7	B8	B9
1	**1**	B2	B3	B4	B5	B6	B7	B8	B9
B2	B2	B4	B5	-4B8		B7		B9	
B3	B3	B5	B6		B7	B8	B9		
B4	B4	-4B8		-4B9					
B5	B5		B7			B9			
B6	B6	B7	B8		B9				
B7	B7		B9						
B8	B8	B9							
B9	B9								

Table 6.5: Multiplication on $\mathscr{H}_1$ when $W = x^3y + y^4$

In the projection $\mathscr{H}_1 \to \mathscr{H}_g$, all basis elements map to 0 but **1**, so any product $\lfloor m \neq 1, 1\rceil \star \lfloor 1, g \neq 1\rceil = 0$ necessarily.

As before, we notice that if $h \neq g^{-1}$, then $\lfloor 1, g\rceil \star \lfloor 1, h\rceil$ lands in the $gh \neq 1$ sector. The basis element here is $\lfloor 1, gh\rceil$, with bi-degree (1/2,1/2). The sum of the degrees of $\lfloor 1, g\rceil$ and $\lfloor 1, h\rceil$ is (1,1). Thus, to preserve the grading, we require that $\lfloor 1, g\rceil \star \lfloor 1, h\rceil = 0$ if $h \neq g^{-1}$. This is captured by the condition that $\alpha_g \star \alpha_h = 0$ unless $V^g + V^h + V^{gh} = V$, as stipulated in 5.2.

If $h = g^{-1}$, then $\lfloor 1, g\rceil \star \lfloor 1, h\rceil$ is determined by the special example worked out toward the end of 5.2; it is $\frac{1}{\langle \lfloor 1, gh\rceil, \lfloor \text{Hess } (gh)^{-1}, (gh)^{-1}\rceil\rangle}\lfloor \text{Hess } gh, gh\rceil$. In this case, this is $12\lfloor xy^3, 1\rceil$.

We can verify that this multiplication satisfies the requirements of a graded Frobenius algebra. Since $\lfloor 1, g\rceil \star \lfloor 1, h\rceil = \begin{cases} 0 & h \neq g^{-1} \\ 1/12\lfloor xy^3, 1\rceil & h = g^{-1} \end{cases}$, the multiplication respects the grading. The multiplication is associative, as can easily be verified by checking on the basis elements.

6.2.1 Invariants.

The invariants are **1** and B9-B16. Again, the multiplication is closed with respect to invariants, so $\mathscr{B}_W^G$ is a Frobenius algebra.

6.3 Modified P_8: $x^3+y^3+z^3+w^2$ with $G \cong S^3$ on the variables

Let $W = x^3 + y^3 + z^3 + w^2$ and $G = S_3$ be the symmetric group on the first three standard basis vectors for $\mathbb{C}^4$. Strictly speaking, this group is inadmissible for a $\mathscr{B}$-model construction; the determinant of the transpositions is negative. The w^2 is added to circumvent this by allowing an extra negative be added to the action of the transpositions, as follows.

$$G = \left\{ 1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, (12) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, (13) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \right.$$
$$\left. (23) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, (123) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, (132) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$$

Every group element here has nontrivial fixed locus. We present the fixed loci, restrictions of W, basis for $\mathscr{H}_g$, and degree of each element in table 6.6. Note the absence of any basis element corresponding to w. The derivative of w^2 is $2w$, so it contributes nothing to any $\mathscr{H}$. In the table we use dashes to indicate when there is no change from the left. The plus and minus gradings are the same in each case, so we only record one.

We now present the nonzero pairings; recall that the pairing of two elements is 0 unless they come from inverse sectors. As demonstrated in Section 5.3, the pairing satisfies the Frobenius property. The pairing on the identity sector is given by the following function:

$$\langle x^a y^b z^c, x^d y^e z^f \rangle = \begin{cases} 1/54 & a+d = 1 = b+e = c+f \\ 0 & \text{otherwise} \end{cases}$$

The pairing of $\mathscr{H}_{(12)}$ is given by the table 6.7; the pairings on $\mathscr{H}_{(13)}$ and $\mathscr{H}_{(23)}$ are identical.

Group Element	1	(12)	(13)	
Fixed Locus (or Basis)	$\mathbb{C}^4$	$\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$	-	
Corresponding Dual Basis	x, y, z, w	r_1, r_2	s_2, s_2	
$W\vert_{V^g}$	W	$2r_1^3 + r_2^3$	$2s_1^3 + s_2^3$	
Basis for $\mathscr{H}_g$	$x^a y^b z^c$ $0 \leq a, b, c \leq 1$	$1, r_1, r_2, r_1 r_2$	$1, s_1, s_2, s_1 s_2$	
Bi-degrees	$\frac{a+b+c}{3}$	$\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{2}$	$\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{2}$	

Group Element	(23)	(123)	(132)	Intersection of 2- and 3-cycle
Fixed Locus Basis	$\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$	-	$\begin{pmatrix}1\\1\\1\\0\end{pmatrix}$
Corresponding Dual Basis	t_1, t_2	u_1, u_2	-	$1, v$
$W\vert_{V^g}$	$2t_1^3 + t_2^3$	$3u_1^3 + u_2^2$	-	$3v^3$
Basis for $\mathscr{H}_g$	$1, t_1, t_2, t_1 t_2$	$1, u_1$	-	$1, v$
Bi-degrees	$\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{2}$	$\frac{1}{3}, \frac{2}{3}$	$\frac{1}{3}, \frac{2}{3}$	-

Table 6.6: Table of State Space Information for $W = x^3 + y^3 + z^3 + w^2$ with group $\cong S_3$.

The pairings between the $\mathscr{H}_{(123)}$ and $\mathscr{H}_{(132)}$ sectors are given by table 6.8

$\mathscr{H}_{(12)}$	1	r_1	r_2	$r_1 r_2$
1				$1/18$
r_1			$1/18$	
r_2		$1/18$		
$r_1 r_2$	$1/18$			

Table 6.7: The pairing for $\mathscr{H}_{(12)}$ (and $\mathscr{H}_{(13)}$ and $\mathscr{H}_{(23)}$).

$\mathscr{H}_{(123)}/\mathscr{H}_{(132)}$	1	u_1
1		$1/18$
u_1	$1/18$	

Table 6.8: The pairing between $\mathscr{H}_{(123)}$ and $\mathscr{H}_{(132)}$.

With the pairings defined, we can introduce the multiplication. As anticipated by Theorem 4.28, there are three types of sectors corresponding to the three conjugacy classes of S_3. We present multiplication by conjugacy class representative, since multiplication of elements of other sectors is similar.

Consider products between $\mathscr{H}_{(12)}$ and $\mathscr{H}_{(12)}$. The common fixed locus is that of (12), so the projection is the identity map. The product is determined by the images of elements in $\mathscr{H}_{(12)}$. We trace them through $\eta^\flat$, $\widehat{f}$, and $\eta^\sharp$:

$$1 \mapsto \left(\alpha \mapsto \begin{cases} 1/18, & \alpha = r_1 r_2 \\ 0, & \text{otherwise} \end{cases}\right) \mapsto \left(\beta \mapsto \begin{cases} 1/18, & \beta = xz, yz \\ 0, & \text{otherwise} \end{cases}\right) \mapsto 1/18 * 54(x+y)$$

$$r_1 \mapsto \left(\alpha \mapsto \begin{cases} 1/18, & \alpha = r_2 \\ 0, & \text{otherwise} \end{cases}\right) \mapsto \left(\beta \mapsto \begin{cases} 1/18, & \beta = z \\ 0, & \text{otherwise} \end{cases}\right) \mapsto 3xy$$

$$r_2 \mapsto \left(\alpha \mapsto \begin{cases} 1/18, & \alpha = r_1 \\ 0, & \text{otherwise} \end{cases}\right) \mapsto \left(\beta \mapsto \begin{cases} 1/18, & \beta = x, y \\ 0, & \text{otherwise} \end{cases}\right) \mapsto 3(x+y)z$$

$$r_1 r_2 \mapsto \left(\alpha \mapsto \begin{cases} 1/18, & \alpha = 1 \\ 0, & \text{otherwise} \end{cases}\right) \mapsto \left(\beta \mapsto \begin{cases} 1/18, & \beta = 1 \\ 0, & \text{otherwise} \end{cases}\right) \mapsto 3xyz$$

We use this to fill out the multiplication table - we take pairs of elements of $\mathscr{H}_{(12)}$, multiply them together and by $\epsilon_{(12)}$, then trace through the diagram. For associativity, we set $\epsilon_{(12)}$

to 1/3. Note that $V^{(12)(12)} = V$, so the diagonal condition would tell us that $\epsilon_{(12)} = 1$. The multiplication between elements of sector $\mathscr{H}_{(12)}$ is summarized in table 6.9. As usual, in this table a blank indicates a 0. We note that this multiplication preserves the bi-grading: the bi-degree of $\lfloor 1, (12)\rceil$ is (1/6,1/6). Two of these multiply to $\lfloor x+y, 1\rceil$, which has bi-degree (1/3,1/3).

$\star$	$\lfloor 1, (12)\rceil$	$\lfloor r_1, (12)\rceil$	$\lfloor r_2, (12)\rceil$	$\lfloor r_1 r_2, (12)\rceil$
$\lfloor 1, (12)\rceil$	$\lfloor x+y, 1\rceil$	xy	$(x+y)z$	xyz
$\lfloor r_1, (12)\rceil$	xy		xyz	
$\lfloor r_2, (12)\rceil$	$(x+y)z$	xyz		
$\lfloor r_1 r_2, (12)\rceil$	xyz			

Table 6.9: Multiplication on the sector $\mathscr{H}_{(12)}$.

Most of the multiplications preserve the bi-grading and associativity with $\epsilon_{g,h}$ a nonzero constant that makes the products monic. The only case where the diagonal condition on $\epsilon_{g,h}$ would set it to 0 is in the case of $\mathscr{H}_{(123)} \star \mathscr{H}_{(123)}$. Here $V^{(123)} + V^{(123)} + V^{(132)} = V^{(123)} \neq V$. To determine the products, we start by tracing elements of $\mathscr{H}_{(123),(123)} = \mathscr{H}_{(123)}$ through the multiplication template. Since $\mathscr{H}_{(123),(123)} \cong \mathscr{H}_{(132)}$, the process is simple and we have $1 \mapsto 1$, $u_1 \mapsto u_1$.

If $\epsilon_{(123),(123)}$ is a nonzero constant c, this would lead to the $\lfloor 1, (123)\rceil \star \lfloor 1, (123)\rceil = c\lfloor 1, (132)\rceil$. Comparing bi-degrees, we have $(1/3, 1/3) + (1/3, 1/3) \neq (1/3, 1/3)$. Clearly $\epsilon_{(123),(123)}$ is not just a constant. If we follow the proscription of the diagonal case and set $\epsilon_{(123),(123)}$ to 0, since $V^{(123)} + V^{(123)} + V^{(132)} \neq V$, we find that most products must be 0 to satisfy associativity. To avoid a trivial multiplication, we must set $\epsilon_{(123),(123)}$ so that 1 maps to something with bi-degree (1/3,1/3). The only option we have is u_1, which clearly satisfies the bi-degree requirement. For associativity, we set $\epsilon_{(123),(123)}$ to $2u_1$. The multiplication table on $\mathscr{H}_{(123)}$ is given in Table 6.10

$\star$	$\lfloor 1, (123)\rceil$	$\lfloor u_1, (123)\rceil$
$\lfloor 1, (123)\rceil$	$2\lfloor u_1, (132)\rceil$	
$\lfloor u_1, (123)\rceil$		

Table 6.10: Multiplication on the sector $\mathscr{H}_{(123)}$.

Proceeding in this way, we develop multiplication tables up to factors of $\epsilon_{g,h}$. A brute force check on triples of the form $\lfloor 1, g \rceil \star \lfloor 1, h \rceil \star \lfloor 1, k \rceil$, as mentioned in Chapter 5, gives a system of equations that helps determine what the ϵ factors should be. Once every equation is satisfied, we have that the multiplication is associative.

We have $\epsilon_{1,g} = 1$ for all g. The factor ϵ between a 2-cycle and itself is 1/3. For different 2-cycles it is 1/2. For 2 and 3-cycles, it's 1/2. For inverse non-identity elements, $\epsilon_{g,g^{-1}} = 1/3$. For products of distinct 2-cycles and products between 2-cycle and 3-cycle sectors, $\epsilon_{g,h} = 1/2$. We have already shown that $\epsilon_{(123),(123)} = \epsilon_{(123),(123)} = 2v_1$. Given these ϵ's, we present the following multiplication tables. We don't write the products of the identity sector with itself, since they are determined in the diagonal case. These, and similar tables for other 2-cycles and 3-cycles, completely determine the multiplication. Calculation on triples of $\lfloor 1, g \rceil$ products verifies the associativity of the multiplication. The multiplication is presented in Table 6.11.

6.3.1 Invariants. We use the isomorphism of Theorem 4.28 and the associated lemmas to determine the invariants of $\mathscr{B}_W^G$ from the centralizer invariants of conjugacy class representatives. We use 1, (12), and (123) as our representatives. The centralizer of identity is the whole group. Using the π operator discussed in the beginning of Section 4.5, we find that the invariants of the identity sector are:

$$\mathrm{Span}_{\mathbb{C}}\{\lfloor 1, 1 \rceil, \lfloor x+y+z, 1 \rceil, \lfloor xy+xz+yz, 1 \rceil, \lfloor xyz, 1 \rceil\}.$$

The centralizer of (12) is $\{1, (12)\}$. Both of these group elements act as the identity on $\mathscr{H}_{(12)}$, so the $C_G(12)$-invariants of the (12) sector are

$$\mathrm{Span}_{\mathbb{C}}\{\lfloor 1, (12) \rceil, \lfloor r_1, (12) \rceil, \lfloor r_2, (12) \rceil, \lfloor r_1 r_2, (12) \rceil\}.$$

The centralizer of (123) is A_3. Again, everything in this group acts as the identity on $\mathscr{H}_{(123)}$,

$\star$	$\mathbf{1}$	$\lfloor x, 1\rceil$	$\lfloor y, 1\rceil$	$\lfloor z, 1\rceil$	$\lfloor xy, 1\rceil$	$\lfloor xz, 1\rceil$	$\lfloor yz, 1\rceil$	$\lfloor xyz, 1\rceil$
$\lfloor 1, 12\rceil$	$\lfloor 1, (12)\rceil$	r_1	r_1	r_2		r_1r_2	r_1r_2	
$\lfloor r_1, (12)\rceil$	r_1			r_1r_2				
$\lfloor r_2, (12)\rceil$	r_2	r_1r_2	r_1r_2					
$\lfloor r_1r_2, (12)\rceil$	r_1r_2							

$\star$	$\lfloor 1, (12)\rceil$	$\lfloor r_1, (12)\rceil$	$\lfloor r_2, (12)\rceil$	$\lfloor r_1r_2, (12)\rceil$
$\lfloor 1, (12)\rceil$	$\lfloor x+y, 1\rceil$	xy	$(x+y)z$	xyz
$\lfloor r_1, (12)\rceil$	xy		xyz	
$\lfloor r_2, (12)\rceil$	$(x+y)z$	xyz		
$\lfloor r_1r_2, (12)\rceil$	xyz			

$\star$	$\lfloor 1, (13)\rceil$	$\lfloor s_1, (13)\rceil$	$\lfloor s_2, (13)\rceil$	$\lfloor s_1s_2, (13)\rceil$
$\lfloor 1, (12)\rceil$	$\lfloor 1, (132)\rceil$	u_1	u_1	
$\lfloor r_1, (12)\rceil$	u_1			
$\lfloor r_2, (12)\rceil$	u_1			
$\lfloor r_1r_2, (12)\rceil$				

$\star$	$\lfloor 1, (123)\rceil$	$\lfloor u_1, (123)\rceil$
$\lfloor 1, (12)\rceil$	$\lfloor t_1+t_2, (23)\rceil$	t_1t_2
$\lfloor r_1, (12)\rceil$	t_1t_2	
$\lfloor r_2, (12)\rceil$	t_1t_2	
$\lfloor r_1r_2, (12)\rceil$		

$\star$	$\mathbf{1}$	$\lfloor x, 1\rceil$	$\lfloor y, 1\rceil$	$\lfloor z, 1\rceil$	$\lfloor xy, 1\rceil$	$\lfloor xz, 1\rceil$	$\lfloor yz, 1\rceil$	$\lfloor xyz, 1\rceil$
$\lfloor 1, (123)\rceil$	$\lfloor 1, 123\rceil$	u_1	u_1	u_1				
$\lfloor u_1, (123)\rceil$	u_1							

$\star$	$\lfloor 1, (12)\rceil$	$\lfloor r_1, (12)\rceil$	$\lfloor r_2, (12)\rceil$	$\lfloor r_1r_2, (12)\rceil$
$\lfloor 1, (123)\rceil$	$\lfloor s_1+s_2, (13)\rceil$	s_1s_2	s_1s_2	
$\lfloor u_1, (123)\rceil$	s_1s_2			

$\star$	$\lfloor 1, (123)\rceil$	$\lfloor u_1, (123)\rceil$
$\lfloor 1, (123)\rceil$	$2\lfloor u_1, (132)\rceil$	
$\lfloor u_1, (123)\rceil$		

$\star$	$\lfloor 1, (132)\rceil$	$\lfloor u_1, (132)\rceil$
$\lfloor 1, (123)\rceil$	$\lfloor xy+yz+xz, 1\rceil$	xyz
$\lfloor u_1, (123)\rceil$	xyz	

Table 6.11: The multiplication of $\mathscr{H}$ for $x^3+y^3+z^3+w^2$.

so the centralizer invariants of $\mathscr{H}_{(123)}$ are the elements of $\mathscr{H}_{(123)}$.

The isomorphism of Theorem 4.28 says that every G-invariant of $\mathscr{H}$ is obtained from a $C_G(g)$ invariant of $\mathscr{H}_g$ by summing over conjugators b of the invariant. That is, if $\lfloor m, g \rceil$ is $C_G(g)$ invariant, and for each a in the conjugacy class $\mathcal{K}$ of g, the element $b_a \in G$ is chosen so that $b_a^{-1} g b_a = a$, then $\sum_{a \in \mathcal{K}} \lfloor m, g \rceil \cdot b_a$.

We obtain the other invariants by applying this process to the centralizer invariants of (12) and (123). A basis for $\mathscr{B}_W^G$ is presented in Table 6.12.

Invariant	$\lfloor 1, 1 \rceil$	$\lfloor x+y+z, 1 \rceil$	$\lfloor xy+xz+yz, 1 \rceil$	$\lfloor xyz, 1 \rceil$
Number	**1**	A2	A3	A4

Invariant	$\lfloor 1, (12) \rceil + \lfloor 1, (13) \rceil + \lfloor 1, (23) \rceil$	$\lfloor r_1, (12) \rceil + \lfloor s_1, (13) \rceil + \lfloor t_1, (23) \rceil$
Number	A5	A6

Invariant	$\lfloor r_2, (12) \rceil + \lfloor s_2, (13) \rceil + \lfloor t_2, (23) \rceil$	$\lfloor r_1 r_2, (12) \rceil + \lfloor s_1 s_2, (13) \rceil + \lfloor t_1 t_2, (23) \rceil$
Number	A7	A8

Invariant	$\lfloor 1, (123) \rceil + \lfloor 1, (132) \rceil$	$\lfloor u_1, (123) \rceil + \lfloor u_1, (132) \rceil$
Number	A9	A10

Table 6.12: The invariants of $\mathscr{H}$ for $W = P_8$; i.e. a basis for $\mathscr{B}_W^G$.

Using the multiplication on the unprojected state space (see Table 6.11), which was already verified to be associative, we present the multiplication of the invariants in Table 6.13.

$\star$	**1**	A2	A3	A4	A5	A6	A7	A8	A9	A10
1	**1**	A2	A3	A4	A5	A6	A7	A8	A9	A10
A2	A2	3A3	3A4		2A6 + A7	A8	2A8		3A9	
A3	A3	3A4			2A8					
A4	A4									
A5	A5	2A6 + A7	2A8		2A2 + 3A9	A3 + 3A10	2A3 + 3A10	3A4	2(A6 + A7)	2A8
A6	A6	A8			A3 + 3A10		3A4		2A8	
A7	A7	2A8			A3 + 3A10	3A4			2A8	
A8	A8				3A4					
A9	A9	3A9			2(A6 + A7)	2A8	2A8		2(A10 + 2A3)	2A4
A10	A10				2A8				2A4	

Table 6.13: The multiplication table of $\mathscr{B}_W^G$ for $W = x^3 + y^3 + z^3 + w^2$ and $G = S_3$.

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